Differential Equations: An Equation in which, the independent, dependent variables and the differential coefficients of dependent variables w.r.t. independent variables are involved is called a differential equation.

Following are a few examples of differential equations.

1. $(3y-4x)\frac{dy}{dx}+(7y^2+6y)=0$ 2. $\frac{d^3y}{dx^3}+\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^4=e^{4x}$

3. $\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}=\frac{d^2y}{dx^2}$ 4. $y=x\left(\frac{dy}{dx}\right)+\frac{1}{\left(\frac{dy}{dx}\right)}$ 5. $\frac{d^2y}{dx^2}=-a^2x$

6. $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=2u$ 7. $\frac{\partial^2 u}{\partial x^2}-a^2\frac{\partial^2 u}{\partial y^2}=0$

Ordinary Differential Equation (O.D.E)

A differential equation is said to be ordinary differential equation, if the derivatives involved in it are w.r.t a single independent variable.

In the above examples equations (1) to (5) are ordinary differential equations.

partial Differential Equation

A differential equation is said to be partial differential equation, if the derivative involved in it are w.r.t more than one independent variable.

In the above examples, the equation (6) and (7) are partial differential equations.

Order & degree of a differential equation:

The order of the D.E is the order of the highest derivative present in the equation and the degree of the D.E is the degree of the highest order derivative after clearing the fractional-powers.

In the above examples

(1) is of order 1 and degree 1

(2) is of order 3 and degree 1

(3) can be written as $\left[1+\left(\frac{dy}{dx}\right)^2\right]^3=\left(\frac{d^2y}{dx^2}\right)^2$ and therefore is of order 2 and degree 2.

(4) can be written as $y\left(\frac{dy}{dx}\right)=x\left(\frac{dy}{dx}\right)^2+1$ and therefore is of order 1 and degree 2.

(5) is of order 2 and degree 1.
(6) is of order 1 and degree 1. (7) is of order 2 and degree 1.

Solution of a differential Equation

A solution of a D.E is a relationship between the dependent and independent variables, which satisfies the given differential equation.

For ex : 1) $y = e^{3x}$ is a solution of the D.E

$$\frac{dy}{dx} - 3e^{3x} = 0$$

$$\frac{d}{dx}(e^{3x}) - 3e^{3x} = 3e^{3x} - 3e^{3x} = 0$$

2) $x = 2\cos\omega t + 3\sin\omega t$ is a solution of the D.E

$$\frac{d^2x}{dt^2} + \omega^2 x = 0$$

$$\frac{d^2}{dt^2}(2\cos\omega t + 3\sin\omega t) + \omega^2(2\cos\omega t + 3\sin\omega t)$$

$$= \frac{d}{dt}(-2\omega\sin\omega t + 3\cos\omega t \cdot \omega) + 2\omega^2\cos\omega t + 3\omega^2\sin\omega t$$

$$= -\cancel{2\omega^2\cos\omega t} + 3\omega^2(-\sin\omega t) + \cancel{2\omega^2\cos\omega t} + 3\omega^2\sin\omega t$$

$$= 0$$

General solution and particular solution

Consider the D.E $\frac{dy}{dx} - 3e^{3x} = 0$ and try to find y. i.e., $\frac{dy}{dx} = 3e^{3x}$ and to find y we need to get rid off $\frac{d}{dx}$ being the differential operator. Obviously we have to use anti differentiation, that is integration

$$\frac{dy}{dx} = 3e^{3x}$$

$$\Rightarrow y = \int 3e^{3x}\,dx + c \quad \text{i.e.,} \quad y = \frac{3e^{3x}}{3} + c$$

or $y = e^{3x} + c$, c being the constant of integration.

Integration is directly or indirectly involved in the process of getting a solution of the given D.E and accordingly the solution will be involved with arbitrary constants. Such a solution is called as the general solution of the D.E. It is obvious that the number of arbitrary constants present in the solution is equal to the order of the D.E.

If the arbitrary constants present in the solution are evaluated by using a set of given conditions then the solution so obtained is called a particular solution.

(or) A solution obtained from the general solution by ③ Giving particular values to the constants involved in it, is called a particular Solution of the equation.

Let us consider the initial condition $y(0)=1$ That is $y=1$ when $x=0$.

The general solution $y=e^{3x}+c$ becomes

$1=e^0+c$ or $1+c=1$ which gives $c=0$.

Thus $y=e^{3x}$ is a particular solution of the D.E.

Solution of Ordinary differential equations of first Order and first degree

First order and first degree equation will be of the form

$$\frac{dy}{dx}=f(x,y) \quad \text{or} \quad M(x,y)\,dx+N(x,y)\,dy=0,$$

We discuss mainly classified four types of differential equations of first order and first degree. They are as follows:

- Variables Separable equations
- Homogeneous equations
- Exact equations
- Linear equations

Variables separable Equations

If the given D.E can be put in the form such that the coefficient of dx is a function of the variable x only and the coefficient of dy is a function of y only then the given equation is said to be in the Separable form. The modified form of such an equation will be,

$P(x)\,dx+Q(y)\,dy=0$. Integrating we have

$$\int P(x)\,dx+\int Q(y)\,dy=c$$

This is the general solution of the equation.

1. Solve : $\sec^2 x \tan y\,dx+\sec^2 y \tan x\,dy=0$

$\sec^2 x \tan y\,dx+\sec^2 y\tan x\,dy=0$

Dividing throughout by $\tan y.\tan x$ we get

$$\frac{\sec^2 x}{\tan x}\,dx+\frac{\sec^2 y}{\tan y}\,dy=0 \quad (\text{variables are separated})$$

$$\Rightarrow \int\frac{\sec^2 x}{\tan x}\,dx+\int\frac{\sec^2 y}{\tan y}\,dy=c \text{ , i.e., } \log(\tan x)+\log(\tan y)=c$$

i.e., $\log(\tan x \cdot \tan y) = \log K$ (say)

Thus $\tan x \cdot \tan y = K$ is the required solution.

2. Solve : $e^x(y-1)\,dx + 2(e^x+4)\,dy = 0$

Soln $e^x(y-1)\,dx + 2(e^x+4)\,dy = 0$

Dividing throughout by $(y-1)(e^x+4)$ we get

$$\frac{e^x}{e^x+4}\,dx + 2\frac{dy}{y-1} = 0 \text{ (variables are separated)}$$

$\Rightarrow$ $\int \frac{e^x}{e^x+4}\,dx + 2\int \frac{dy}{y-1} = c$ i.e., $\log(e^x+4) + 2\log(y-1) = c$

$\log(e^x+4) + \log(y-1)^2 = c \Rightarrow \log[(e^x+4)(y-1)^2] = \log K$ (say)

Thus $(e^x+4)(y-1)^2 = K$ is the required solution.

3. Solve : $\frac{dy}{dx} = \frac{x(2\log x+1)}{\sin y + y\cos y}$

$\rightarrow$ The given equation can be put in the form

$(\sin y + y\cos y)\,dy = (2x\log x + x)\,dx$ by separating the variables

$\Rightarrow$ $\int \sin y\,dy + \int y\cos y\,dy - 2\int x\log x\,dx - \int x\,dx = c$

on integration we get,

$$-\cos y + \left\{y\sin y - \int \sin y \cdot 1\,dy\right\} - 2\left\{\log x \cdot \frac{x^2}{2} - \int \frac{x^{\cancel{2}}}{2}\cdot\frac{1}{\cancel{x}}\,dx\right\} - \frac{x^2}{2} = c$$

i.e., $-\cancel{\cos y} + y\sin y + \cancel{\cos y} - x^2\log x + \cancel{\frac{x^2}{2}} - \cancel{\frac{x^2}{2}} = c$

i.e., $y\sin y - x^2\log x = c$ is the required solution

4. Solve: $\frac{dy}{dx} = xe^{y-x^2}$ given that $y(0)=0$.

$\Rightarrow$ $\frac{dy}{dx} = xe^{y-x^2}$ or $\frac{dy}{dx} = x\,e^y \cdot e^{-x^2}$

i.e., $\frac{dy}{e^y} = xe^{-x^2}dx$ by separating the variables

$\Rightarrow$ $\int e^{-y}dy = \int xe^{-x^2}dx + c \Rightarrow \int e^{-y}dy - \int xe^{-x^2}dx = c$

i.e., $-e^{-y} - \int xe^{-x^2}dx = c$

put $-x^2 = t \therefore -2x\,dx = dt$ or $-x\,dx = dt/2$

Hence we have, $-e^{-y} + \int e^t\,dt/2 = c$

i.e., $-e^{-y} + e^t/2 = c$

or $\frac{e^{-x^2}}{2} - e^{-y} = c$ is the general solution

Now we consider $y(0)=0$ That is $y=0$ when $x=0$.

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The general solution becomes

$$\frac{1}{2}-1=C \quad \text{or } C=-1/2$$

Now the general solution becomes

$$\frac{e^{-x^2}}{2}-e^{-y}=-1/2.$$

Thus $(e^{-x^2}+1)=2e^{-y}$ is the particular solution.

5. Solve: $\frac{dy}{dx}=e^{3x-2y}+x^2e^{-2y}$

Soln $\frac{dy}{dx}=e^{3x-2y}+x^2e^{-2y}$

i.e., $\frac{dy}{dx}=e^{-2y}(e^{3x}+x^2)$ or $\frac{dy}{e^{-2y}}=(e^{3x}+x^2)dx$ by separating the variables

$\Rightarrow \int e^{2y}dy-\int(e^{3x}+x^2)dx=C$

Thus $\frac{e^{2y}}{2}-\frac{e^{3x}}{3}-\frac{x^3}{3}=C$, is the required solution.

6. Solve: $(xy-x)\,dx+(xy+y)\,dy=0$

Soln $(xy-x)\,dx+(xy+y)\,dy=0$

or $x(y-1)\,dx+y(x+1)\,dy=0$ Dividing throughout by $(y-1)(x+1)$ we get

$\frac{x}{x+1}dx+\frac{y}{y-1}dy=0$ (Variables are separated)

$$\Rightarrow \int\frac{x}{x+1}dx+\int\frac{y}{y-1}dy=C$$

or $\int\frac{(x+1)-1}{x+1}dx+\int\frac{(y-1)+1}{y-1}dy=C$

i.e., $\int 1\,dx-\int\frac{1}{x+1}dx+\int 1\,dy+\int\frac{1}{y-1}dy=C$

i.e., $x-\log(x+1)+y+\log(y-1)=C$

Thus $(x+y)+\log\left(\frac{y-1}{x+1}\right)=C$, is the required solution

7. Solve: $xy\frac{dy}{dx}=1+x+y+xy$

Soln $xy\frac{dy}{dx}=1+x+y+xy=(1+x)+y(1+x)$

i.e., $xy\frac{dy}{dx}=(1+x)(1+y)$

or $\frac{y}{1+y}dy=\frac{1+x}{x}dx$ by separating the variables

$\Rightarrow \int\frac{y}{1+y}dy-\int\frac{1+x}{x}dx=C$ or $\int\frac{(1+y)-1}{1+y}dy-\int\frac{1}{x}dx-\int 1\,dx=C$

i.e., $\int 1\,dy-\int\frac{1}{1+y}dy-\int\frac{1}{x}dx-\int 1\,dx=C$

i.e., $y-\log(1+y)-\log x-x=C$

Thus $(y-x)-\log[x(1+y)]=C$ is the required solution.

8. Solve: $(xy+x)\,dx + (x^2y^2+x^2+y^2+1)\,dy=0$

Soln $(xy+x)dx + (x^2y^2+x^2+y^2+1)\,dy=0$

or $x(y+1)dx + (x^2(y^2+1)+1(y^2+1))dy=0$

i.e., $x(y+1)dx + (x^2+1)(y^2+1)dy=0$

Dividing throughout by $(y+1)(x^2+1)$ we get,

$$\frac{x}{x^2+1}dx + \frac{y^2+1}{y+1}dy=0 \text{ (variables are separated)}$$

$$\Rightarrow \int\frac{x}{x^2+1}dx + \int\frac{y^2+1}{y+1}dy = c$$

or $$\frac{1}{2}\int\frac{2x}{x^2+1}dx + \int\frac{(y^2-1)+2}{y+1}dy=c$$

i.e., $$\frac{1}{2}\log(x^2+1) + \int\frac{(y-1)(y+1)}{y+1}dy + 2\int\frac{dy}{y+1}=c$$

i.e., $$\frac{1}{2}\log(x^2+1) + \int(y-1)\,dy + 2\log(y+1)=c$$

$$\log\sqrt{x^2+1} + \frac{y^2}{2}-y + \log(y+1)^2 = c$$

Thus, $\log\left[\sqrt{x^2+1}\cdot(y+1)^2\right] + \frac{y^2}{2}-y=c$, is the required solution.

9. Solve: $(1-y)x\frac{dy}{dx} + (1+x)y=0$

Soln $(1-y)x\frac{dy}{dx} + (1+x)y=0 \div xy$

$$\frac{1-y}{y}\frac{dy}{dx} + \frac{1+x}{x}=0$$

$$\frac{1-y}{y}dy + \frac{1+x}{x}dx=0$$

$$\int\frac{1-y}{y}dy + \int\frac{1+x}{x}dx=0$$

$$\int\left(\frac{1}{y}-1\right)dy + \int\left(\frac{1}{x}+1\right)dx=0 \Rightarrow \log y - y + \log x + x = c.$$

$\Rightarrow \log(xy)+x-y=c$ is the required solution

10. Solve $\frac{dy}{dx} + \frac{1+\cos 2y}{1+\cos 2x}=0.$

Soln

$$\frac{dy}{dx} = -\frac{1+\cos 2y}{1+\cos 2x}$$

$$\frac{dy}{1+\cos 2y} = \frac{-dx}{1+\cos 2x}$$

$1+\cos 2x = 2\cos^2 x$

$1+\cos 2y = 2\cos^2 y$

$$\frac{dy}{2\cos^2 y} = -\frac{1}{2\cos^2 x}dx.$$

$$\frac{1}{2}\int\sec^2 y\,dy + \frac{1}{2}\int\sec^2 x\,dx = c \Rightarrow \tan y + \tan x = 2c = c'$$

$$\tan y + \tan x = c'.$$

Equations reducible to the variables separable form

Some differential equations can be reduced to the variables separable form by taking a suitable substitution. We identify a few types of equation along with the associated substitutions.

(i) $\frac{dy}{dx} = f(ax+by+c)$; Substitution : $t = ax+by+c$

(ii) $\frac{dy}{dx} = \frac{(ax+by)+c}{K(ax+by)+c'}$; Substitution : $t = ax+by$

① Solve $\frac{dy}{dx} = (9x+y+1)^2$

Soln we have $\frac{dy}{dx} = (9x+y+1)^2 \rightarrow (1)$

put $t = 9x+y+1 \therefore \frac{dt}{dx} = 9+\frac{dy}{dx}$ or $\frac{dt}{dx} - 9 = \frac{dy}{dx}$

Now (1) becomes $\frac{dt}{dx} - 9 = t^2$ or $\frac{dt}{dx} = t^2+9$

Hence $\int \frac{1}{t^2+9} dt = \int dx + c$ i.e., $\int \frac{1}{t^2+3^2} dt = \int dx + c$

$\frac{1}{3}\tan^{-1}\left(\frac{t}{3}\right) - x = c$ Thus $\frac{1}{3}\tan^{-1}\left(\frac{t}{3}\right) - x = c$ $\left[\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + c\right]$

$\Rightarrow \frac{1}{3}\tan^{-1}\left(\frac{9x+y+1}{3}\right) - x = c$, is the required solution.

② Solve: $\cos(x+y+1)\,dx - dy = 0$

or $\frac{dy}{dx} = \cos(x+y+1)$

Soln We have $\frac{dy}{dx} = \cos(x+y+1) \quad \ldots (1)$

Put $t = x+y+1 \therefore \frac{dt}{dx} = 1+\frac{dy}{dx}$ or $\frac{dt}{dx} - 1 = \frac{dy}{dx}$

Now (1) becomes $\frac{dt}{dx} - 1 = \cos t$ or $\frac{dt}{dx} = 1+\cos t$

i.e., $\frac{dt}{1+\cos t} = dx \Rightarrow \int \frac{dt}{1+\cos t} - \int dx = c$

i.e., $\int \frac{dt}{2\cos^2(t/2)} - x = c$ or $\int \frac{1}{2}\sec^2(t/2)\,dt - x = c$

i.e., $\frac{1}{2} \cdot \frac{1}{1/2}\tan(t/2) - x = c$. But $t = x+y+1$ $\left[\int \sec^2 ax\,dx = \frac{1}{a}\tan ax + c\right]$

Thus $\tan\left(\frac{x+y+1}{2}\right) - x = c$ is the required solution.

③ Solve: $\sec(x+y) = dy/dx$

Soln $\frac{dy}{dx} = \sec(x+y) \rightarrow ①$ put $x+y = t$ $\therefore 1+\frac{dy}{dx} = \frac{dt}{dx}$

or $\frac{dt}{dx} - 1 = dy/dx$

Now (1) becomes $\frac{dt}{dx}-1 = \sec t$ or $\frac{dt}{dx} = 1+\sec t$

i.e., $\int \frac{dt}{1+\sec t} - \int dx = c \Rightarrow \int \frac{dt}{1+\frac{1}{\cos t}} - x = c$

$\Rightarrow \int \frac{\cos t}{1+\cos t}\,dt - x = c \Rightarrow \int \frac{(1+\cos t)-1}{1+\cos t}\,dt - x = c$

$\Rightarrow \int 1\,dt - \int \frac{1}{1+\cos t}\,dt - x = c$ or $t - \int \frac{1}{2\cos^2(t/2)}\,dt - x = c$

i.e., $t - \frac{1}{2}\int \sec^2(t/2)\,dt - x = c$

$t - \frac{1}{\cancel{2}}\frac{1}{1/\cancel{2}}\tan(t/2) - x = c$ But $t = x+y$

Thus $\cancel{x}+y - \tan\left(\frac{x+y}{2}\right) - \cancel{x} = c$

$y - \tan\left(\frac{x+y}{2}\right) = c$, is the required solution

④ solve: $\frac{dy}{dx} = (x+y)^2 + 4(x+y)+6$

Soln putting $x+y=t$, we have $1+\frac{dy}{dx} = \frac{dt}{dx}$

the given equation becomes

$\frac{dt}{dx} - 1 = t^2+4t+6$ or $\frac{dt}{dx} = t^2+4t+7$.

i.e., $\frac{dt}{t^2+4t+7} = dx \Rightarrow \int \frac{dt}{t^2+4t+7} - \int dx = c$

$\therefore \int \frac{dt}{(t+2)^2+(\sqrt{3})^2} - x = c$

Now t^2+4t+7
$= t^2+4t+4+3$
$= t^2+4t+4+3$
$= (t+2)^2+3$

i.e., $\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{t+2}{\sqrt{3}}\right) - x = c$, where $t = x+y$

Thus $\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x+y+2}{\sqrt{3}}\right) - x = c$, is the required solution.

⑤ solve: $(x+2y)(dx-dy) = (dx+dy) \rightarrow (1)$

Soln Rearranging the terms in the given equation we have,

$(x+2y-1)\,dx = (x+2y+1)\,dy$.

or $\frac{dy}{dx} = \frac{(x+2y)-1}{(x+2y)+1}$ put $t = x+2y \Rightarrow \frac{dt}{dx} = 1+2\frac{dy}{dx}$

$\frac{1}{2}\left(\frac{dt}{dx}-1\right) = \frac{dy}{dx}$

Now (1) becomes $\frac{1}{2}\left(\frac{dt}{dx}-1\right) = \frac{t-1}{t+1}$

$\therefore \frac{dt}{dx} = \frac{2(t-1)}{t+1}+1$ or $\frac{dt}{dx} = \frac{2t-2+t+1}{t+1} = \frac{3t-1}{t+1}$

i.e., $\frac{t+1}{3t-1}\,dt = dx \Rightarrow \int \frac{t+1}{3t-1}\,dt - \int dx = c$

i.e., $\int \frac{t+1}{3t-1}\,dt - x = c$

Let $t+1 = l(3t-1)+m$ or $t+1 = 3tl + (-l+m)$

$\Rightarrow 3l = 1$ and $-l+m = 1$

Hence $\boxed{l = 1/3}$ and $-1/3 + m = 1$ or $\boxed{m = \frac{4}{3}}$

$\therefore \; t+1 = \frac{1}{3}(3t-1) + 4/3$

Thus (2) can be written as

$$\boxed{\int \frac{1}{ax+b}\,dx = \frac{1}{a}\log(ax+b) + c}$$

$$\int \frac{1/3(3t-1) + 4/3}{3t-1}\,dt - x = c$$

i.e., $\frac{1}{3}\int 1\,dt + \frac{4}{3}\int \frac{dt}{3t-1} - x = c$ i.e., $\frac{1}{3}t + \frac{4}{3\times 3}\log(3t-1) - x = c$ where $t = x+2y$

i.e., $\frac{x+2y}{3} + \frac{4}{9}\log(3x+6y-1) - x = c$

Thus $\frac{2}{3}(y-x) + \frac{4}{9}\log(3x+6y-1) = c$, is the required Solution.

(6) Solve: $(x-3y+4)\,dy = (2x-6y+7)\,dx \rightarrow (1)$

$$\boxed{\int \frac{ax+b}{cx+d}\,dx = \frac{ax}{c} - \left(\frac{ad-bc}{c^2}\right)\log(cx+d) + c}$$

soln we have $\frac{dy}{dx} = \frac{2x-6y+7}{x-3y+4} = \frac{2(x-3y)+7}{(x-3y)+4}$

put $t = x-3y$ $\therefore \frac{dt}{dx} = 1 - 3\frac{dy}{dx}$ or $-\frac{1}{3}\left(\frac{dt}{dx} - 1\right)$

Now (1) becomes, $-\frac{1}{3}\left(\frac{dt}{dx} - 1\right) = \frac{2t+7}{t+4}$

i.e., $\frac{dt}{dx} = \frac{-6t-21}{t+4} + 1$ i.e., $\frac{dt}{dx} = \frac{-6t-21+t+4}{t+4} = \frac{-5t-17}{t+4}$

i.e., $\frac{dt}{dx} = -\frac{(5t+17)}{t+4} \Rightarrow \int \frac{t+4}{5t+17}\,dt + \int dx = c$

Now, Let $t+4 = l(5t+17) + m \rightarrow (2)$

or $t+4 = 5lt + 17l + m$

$$\boxed{\int \frac{ax+b}{cx+d}\,dx = \frac{ax}{c} - \left(\frac{ad-bc}{c^2}\right)\log(cx+d) + c}$$

$\Rightarrow 5l = 1$ and $17l + m = 4$

Hence $\boxed{l = 1/5}$ and $\frac{17}{5} + m = 4$ or $\boxed{m = 3/5}$

$\therefore \; t+4 = \frac{1}{5}(5t+17) + \frac{3}{5}$

Thus (2) now becomes,

$$\int \frac{\frac{1}{5}(5t+17) + 3/5}{(5t+17)}\,dt + \int dx = c \text{ i.e., } \frac{1}{5}\int 1\,dt + \frac{3}{5}\int \frac{dt}{(5t+17)} + x = c$$

i.e., $\frac{1}{5}t + \frac{3}{25}\log(5t+17) + x = c$, where $t = x-3y$

i.e., $\frac{x-3y}{5} + \frac{3}{25}\log(5x-15y+17) + x = c$

Thus. $\frac{3}{5}(2x-y) + \frac{3}{25}\log(5x-15y+17) = c$, is the required Solution

<u>Assignments</u>

(1) Solve : $\frac{dy}{dx} = \frac{x+y+1}{2x+2y+3}$ Ans: $\frac{1}{3}(2y-x) + \frac{1}{9}\log(3x+3y+4) = c$.

(2) solve : $(x+y+1)^2\frac{dy}{dx} = 1$ Ans: $(y+1) - \tan^{-1}(x+y+1) = c$

Homogeneous Equations:-

Homogeneous function :- A function $f(x,y)$ is said to be homogeneous of degree n in x and y if $f(x,y) = x^n g(y/x)$.

A differential equation of the form,

$$\frac{dy}{dx} = \frac{f(x,y)}{g(x,y)}$$

where $f(x,y)$ and $g(x,y)$ are homogeneous functions of the same degree in x & y is called a Homogeneous differential equations of first order.

Consider a Homogeneous differential equation

$$\frac{dy}{dx} = \frac{f(x,y)}{g(x,y)} \quad \ldots (1)$$

Let $f(x,y)$ and $g(x,y)$ be homogeneous functions of degree n. Then we can write the equation (1)

$$\frac{dy}{dx} = \frac{x^n \phi(y/x)}{x^n \psi(y/x)} \Rightarrow \frac{dy}{dx} = \frac{\phi(y/x)}{\psi(y/x)} \quad \ldots (2)$$

To solve this equation, hence (1), we put

$$y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$$

Then (2) becomes

$$v + x\frac{dv}{dx} = \frac{\phi(v)}{\psi(v)}$$

which can be solved by separating the variables.

① Solve $(x-y)\,dy - (2x-y)\,dx = 0$

Soln The given equation can be written as

$$\frac{dy}{dx} = \frac{2x-y}{x-y}$$

This is a homogeneous equation - for $2x-y$ & $x-y$ are homogeneous equations of degree 1.

put, $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$.

Thus the equation becomes

$$v + x\frac{dv}{dx} = \frac{2x - vx}{x - vx}$$

$$v + x\frac{dv}{dx} = \frac{2-v}{1-v}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{2-v}{1-v} - v = \frac{2-v-v+v^2}{1-v}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{v^2-2v+2}{1-v}$$

By separating the variables, we get

$$\frac{1-v}{v^2-2v+2}\,dv = \frac{dx}{x}$$

$$\Rightarrow \int \frac{1-v}{v^2-2v+2}\,dv = \int \frac{dx}{x} + \log c \quad \left[\int \frac{f'(x)}{f(x)}dx = \log f(x)\right]$$

$$\Rightarrow -\frac{1}{2}\int \frac{2v-2}{v^2-2v+2}\,dv = \int \frac{dx}{x} + \log c$$

$$\Rightarrow -\frac{1}{2}\log(v^2-2v+2) = \log x + \log c$$

$$\Rightarrow 2\log x + \log(v^2-2v+2) + 2\log c = 0$$

$$\log\left(x^2(v^2-2v+2)c^2\right) = 0$$

$$x^2(v^2-2v+2)c^2 = e^0 = 1 \quad [\text{since } v = y/x]$$

$$x^2\left(\left(\frac{y}{x}\right)^2 - 2(y/x) + 2\right)c^2 = 1$$

$$y^2 - 2xy + 2x^2 = 1/c^2$$

$$2x^2 - 2xy + y^2 = c' \quad \text{where } \frac{1}{c^2} = c'$$

2. Solve $(y^2+2xy)dx + (2x^2+3xy)dy = 0$

Soln:- The given equation can be written as

$$\frac{dy}{dx} = -\left[\frac{y^2+2xy}{2x^2+3xy}\right]$$

This is a homogeneous equation.

put $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

Thus the equation becomes

$$v + x\frac{dv}{dx} = -\left[\frac{v^2x^2+2x^2v}{2x^2+3xy}\right]$$

$$\Rightarrow v + x\frac{dv}{dx} = -\left[\frac{v^2+2v}{2+3v}\right]$$

$$\Rightarrow -x\frac{dv}{dx} = v + \frac{v^2+2v}{2+3v} = \frac{2v+3v^2+v^2+2v}{2+3v}$$

$$\Rightarrow -x\frac{dv}{dx} = \frac{4v^2+4v}{2+3v}$$

On separating the variables, we get

$$\frac{2+3v}{4v^2+4v} = -\frac{dx}{x}$$

$$\frac{2+3v}{v^2+v} = -4\frac{dx}{x} \Rightarrow \int\frac{2+3v}{v^2+v}dv + 4\int\frac{dx}{x} = \log c \quad \to (1)$$

Let $\frac{2+3v}{v^2+v} = \frac{2+3v}{v(v+1)} = \frac{A}{v} + \frac{B}{v+1}$

$2+3v = A(v+1) + Bv$

$v=0 \Rightarrow 2 = A$, $v=-1 \Rightarrow -1 = -B$ i.e. $B=1$

$$\therefore \frac{2+3v}{v^2+v} = \frac{2}{v} + \frac{1}{v+1}$$

$$\Rightarrow \int\frac{2+3v}{v^2+v}dv = \int\frac{2\,dv}{v} + \int\frac{1\,dv}{v+1} = 2\log v + \log(v+1) = \log(v^2(v+1))$$

$\therefore$ (1) becomes

$$\log v^2(v+1) + 4\log x = \log c \Rightarrow \log x^4v^2(v+1) = \log c$$

$$\Rightarrow x^4v^2(v+1) = c \Rightarrow x^4\frac{y^2}{x^2}(y/x+1) = c$$

$$\Rightarrow x^2y^2\left(\frac{y}{x}+1\right) = c \quad \text{or} \quad xy^2(x+y) = c.$$

This is the required solution

3. Solve $(x^2+2y^2)dx - xydy = 0$

Soln The given equation can be written as

$$\frac{dy}{dx} = \frac{x^2+2y^2}{xy}$$ This is a homogeneous equation

put $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

Thus the equation becomes,

$$v + x\frac{dv}{dx} = \frac{x^2+2v^2x^2}{x^2v} \Rightarrow v + x\frac{dv}{dx} = \frac{1+2v^2}{v}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{1+2v^2}{v} - v = \frac{1+v^2}{v}$$

On separating the variables, we get

$$\frac{v}{1+v^2}dv = \frac{dx}{x} \Rightarrow \frac{1}{2}\int\frac{2v}{1+v^2}dv = \int\frac{dx}{x} + \log c$$

$$\Rightarrow \frac{1}{2}\log(1+v^2) = \log x + \log c$$

$$\Rightarrow \log(1+v^2) = 2\log xc$$

$$\Rightarrow \log(1+v^2) = \log x^2c^2$$

$$\Rightarrow 1+v^2 = x^2c^2 \Rightarrow 1 + y^2/x^2 = x^2c^2$$

$$\Rightarrow x^2+y^2 = x^4c' \quad \text{where } c^2 = c'$$

4. Solve: $\frac{dy}{dx} = \frac{x^2+y^2}{2xy}$.

Soln This is a homogeneous equation

put $y = vx$ $\therefore \frac{dy}{dx} = v + x\frac{dv}{dx}$

Thus the given equation becomes

$$v + x\frac{dv}{dx} = \frac{x^2+v^2x^2}{2vx^2} = \frac{1+v^2}{2v}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{1+v^2}{2v} - v = \frac{1+v^2-2v^2}{2v} = \frac{1-v^2}{2v}$$

$\therefore$ $\frac{2v}{1-v^2}dv = \frac{1}{x}dx$, by separating the variables

$$\Rightarrow \int \frac{2v}{1-v^2}dv = \int \frac{dx}{x} + \log c$$

$$\Rightarrow -\int \frac{-2v}{1-v^2} - \int \frac{dx}{x} = \log c$$

$$\Rightarrow -\log(1-v^2) - \log x = \log c$$

$$\Rightarrow -[\log(1-v^2) + \log x] = \log c$$

$$\Rightarrow -[\log(1-v^2)x] = \log c$$

$$\Rightarrow \log c + \log(1-v^2)x = 0$$

$$\Rightarrow \log[cx(1-v^2)] = 0 \qquad [\text{since } v = y/x]$$

$$\Rightarrow cx(1-v^2) = 1 \Rightarrow cx(1-y^2/x^2) = 1$$

$$\Rightarrow c\cancel{x}\left(\frac{x^2-y^2}{x^{\cancel{2}}}\right) = 1 \Rightarrow \boxed{c(x^2-y^2) = x}$$

5. Solve: $x\,dy - y\,dx = \sqrt{x^2+y^2}\,dx$

Soln we have $x\,dy = (y + \sqrt{x^2+y^2})dx$

$$\frac{dy}{dx} = \frac{y+\sqrt{x^2+y^2}}{x} \rightarrow (1) \qquad \left[\int \frac{dx}{\sqrt{x^2+a^2}} = \sinh^{-1}\left(\frac{x}{a}\right)\right]$$

put $y = vx$ $\therefore \frac{dy}{dx} = v + x\frac{dv}{dx}$

Now (1) becomes $v + x\frac{dv}{dx} = \frac{vx + \sqrt{x^2+v^2x^2}}{x}$

i.e., $v + x\frac{dv}{dx} = \frac{x[v+\sqrt{1+v^2}]}{x}$ or $x\frac{dv}{dx} = \sqrt{1+v^2}$

$$\therefore \frac{dv}{\sqrt{1+v^2}} = \frac{dx}{x} \Rightarrow \int \frac{dv}{\sqrt{1+v^2}} = \int \frac{dx}{x} + c$$

$$\Rightarrow \sinh^{-1}v = \log x + c \Rightarrow \sinh^{-1}(y/x) - \log x = c$$

is the required solution $\quad [\text{since } v = \frac{y}{x}]$

6. Solve: $[x\tan(y/x) - y\sec^2(y/x)]dx + x\sec^2(y/x)dy = 0$

Soln The given equation can be written as

$$\frac{dy}{dx} = \frac{y\sec^2(y/x) - x\tan(y/x)}{x\sec^2(y/x)} \quad -(1)$$

put $\frac{y}{x} = v$ or $y = vx$ $\therefore \frac{dy}{dx} = v + x\frac{dv}{dx}$

Now (1) becomes, $v + x\frac{dv}{dx} = \frac{vx\sec^2 v - x\tan v}{x\sec^2 v}$

i.e., $v + x\frac{dv}{dx} = \frac{\cancel{x}(v\sec^2 v - \tan v)}{\cancel{x}\sec^2 v}$

i.e., $x\frac{dv}{dx} = \frac{v\sec^2 v - \tan v}{\sec^2 v} - v$

$$x\frac{dv}{dx} = \frac{v\sec^2 v - \tan v - v\sec^2 v}{\sec^2 v}$$

$$x\frac{dv}{dx} = \frac{-\tan v}{\sec^2 v} \quad \text{or} \quad \frac{\sec^2 v}{\tan v}\,dv = -\frac{dx}{x}$$

Hence $\int \frac{\sec^2 v}{\tan v}\,dv + \int \frac{dx}{x} = c$

i.e., $\log(\tan v) + \log x = c = \log k$

$\Rightarrow$ $x\tan v = k$ where $v = y/x$.

Thus $x\tan(y/x) = k$, is the required solution.

7. Solve : $(x\cos y/x + y\sin y/x)y - (y\sin y/x - x\cos y/x)x\frac{dy}{dx} = 0$

Soln $\frac{dy}{dx} = \frac{y(x\cos y/x + y\sin y/x)}{x(y\sin y/x - x\cos y/x)}$

clearly the equation is of homogeneous type.

put $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

$$v + x\frac{dv}{dx} = \frac{vx(x\cos v + vx\sin v)}{x(vx\sin v - x\cos v)}$$

$$v + x\frac{dv}{dx} = \frac{v(\cos v + v\sin v)}{(v\sin v - \cos v)}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{v\cos v + v^2\sin v}{v\sin v - \cos v} - v$$

$$\Rightarrow x\frac{dv}{dx} = \frac{v\cos v + \cancel{v^2\sin v} - \cancel{v^2\sin v} + v\cos v}{v\sin v - \cos v}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{2v\cos v}{v\sin v - \cos v}$$

$$\Rightarrow \frac{v\sin v - \cos v}{2v\cos v}\,dv = \frac{dx}{x} \Rightarrow \frac{v\sin v - \cos v}{v\cos v}\,dv = \frac{2\,dx}{x}$$

$$\Rightarrow -\int \frac{v\sin v - \cos v}{v\cos v}\,dv = 2\int \frac{dx}{x} + \log c$$

$\Rightarrow -\log(v\cos v) = 2\log x + \log c$.

$\Rightarrow \log x^2 + \log c + \log(v\cos v) = 0$

$\Rightarrow \log(x^2 c\, v\cos v) = 0 \Rightarrow x^2 c\, v\cos v = 1$ [since $v = y/x$]

$\Rightarrow xy\cos y/x = c'$ where $c' = 1/c$

8) Solve $x\sin(y/x)dy = (y\sin(y/x) - x)dx$.

Soln

$$\frac{dy}{dx} = \frac{y\sin(y/x) - x}{x\sin(y/x)}$$

put $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

$$v + x\frac{dv}{dx} = \frac{vx\sin v - x}{x\sin v} = \frac{v\sin v - 1}{\sin v}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{v\sin v - 1}{\sin v} - v = \frac{\cancel{v\sin v} - 1 - \cancel{v\sin v}}{\sin v}$$

$$\Rightarrow \int \sin v\, dv = \int -\frac{1}{x}dx + \log c$$

$-\cos v = -\log x + \log c$ $\Rightarrow -\cos v = -\log x + c$

$\log x - \cos v = \log c$ or $\Rightarrow \log x - \cos v = c$

$\log x - \cos y/x = \log c$. $\Rightarrow \log x - \cos(y/x) = c$

9) Solve $(x^2 - 2xy - y^2)dx - (x+y)^2 dy = 0$.

Soln

$$\frac{dy}{dx} = \frac{x^2 - 2xy - y^2}{x^2 + 2xy + y^2} \quad \text{---(1)}$$

Let $y = vx \Rightarrow dy/dx = v + x\frac{dv}{dx}$

$$v + x\frac{dv}{dx} = \frac{x^2 - 2vx^2 - v^2x^2}{x^2 + 2x + 2vx^2 + v^2x^2}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{1 - 2v - v^2}{1 + 2v + v^2} - v$$

$$\Rightarrow x\frac{dv}{dx} = \frac{1 - 2v - v^2 - v - 2v^2 - v^3}{1 + 2v + v^2}$$

$$\Rightarrow x\frac{dv}{dx} = -\frac{(v^3 + 3v^2 + 3v - 1)}{1 + 2v + v^2}$$

$$\Rightarrow \frac{v^2 + 2v + 1}{v^3 + 3v^2 + 3v - 1} = -\frac{dx}{x}$$

$$\Rightarrow \left(\frac{1}{3}\right)\int \frac{3v^2 + 6v + 3}{v^3 + 3v^2 + 3v - 1} = -\int \frac{dx}{x} + \log c$$

$$\Rightarrow \frac{1}{3}\log(v^3 + 3v^2 + 3v - 1) = -\log x + \log c$$

$$\Rightarrow \log(v^3 + 3v^2 + 3v - 1) = -3\log x + 3\log c$$

$$\Rightarrow \log(v^3 + 3v^2 + 3v - 1) + 3\log x = 3\log c$$

$$\Rightarrow \log (v^3+3v^2+3v-1)x^3 = \log c^3$$
$$\Rightarrow (v^3+3v^2+3v-1)x^3 = c^3 \quad [\text{since } v = y/x]$$
$$\Rightarrow \left(\frac{y^3}{x^3}+\frac{3y^2}{x^2}+\frac{3y}{x}-1\right)x^3 = c^3$$
$$\Rightarrow \frac{1}{\cancel{x^3}}(y^3+3xy^2+3x^2y-x^3)\cancel{x^3} = c^3$$
$c^3 = c'$
$$\Rightarrow y^3+3xy^2+3x^2y-x^3 = c'$$
$$\text{or} \quad x^3-3x^2y-3xy^2-y^3 = c'$$

(10) $(1+e^{x/y})dx + e^{x/y}(1-x/y)dy = 0$

soln As we observe terms with x/y, we need to express the equation relating to dx/dy and the terms are homogeneous functions of degree 0.

We have $(1+e^{x/y})dx = e^{x/y}\left(\frac{x}{y}-1\right)dy$

or $$\frac{dx}{dy} = \frac{e^{x/y}(x/y-1)}{1+e^{x/y}}$$

Put $x/y = v$ or $x = vy$ $\therefore \frac{dx}{dy} = v + y\frac{dv}{dy}$

Now (1) becomes, $v + y\frac{dv}{dy} = \frac{e^v(v-1)}{1+e^v}$

i.e., $y\frac{dv}{dy} = \frac{e^v(v-1)}{1+e^v} - v$

i.e., $y\frac{dv}{dy} = \frac{\cancel{e^v v}-e^v-v-\cancel{e^v v}}{1+e^v} = \frac{-(e^v+v)}{1+e^v}$

or $\frac{1+e^v}{e^v+v}dv = -\frac{dy}{y}$

Hence $\int\frac{(1+e^v)}{e^v+v}dv + \int\frac{dy}{y} = c$

i.e., $\log(e^v+v) + \log y = \log k$ (say)

or $\log(e^v+v)y = \log k$

$\Rightarrow (e^v+v)y = k$, where $v = x/y$

Thus $ye^{x/y} + x = k$, is the required solution.

Assignments

Solve the following equations:

1) $\frac{dy}{dx} = \frac{x^2+y^2}{x(x+y)}$ soln $(x-y)^2 = cxe^{-y/x}$

2) $dy/dx - \sin(y/x) = y/x$ soln → $\operatorname{cosec}(y/x) - \cot(y/x) = kx$.

3) $x\frac{dy}{dx} + \frac{y^2}{x} = y$ soln : $\log x - x/y = c$

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(11) Solve $x(x+y)dy - y^2dx = 0$.

Soln The given equation can be written as

$$\frac{dy}{dx} = \frac{y^2}{x(x+y)}$$

Put $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

Thus the equation becomes

$$v + x\frac{dv}{dx} = \frac{v^2x^2}{x(x+vx)} = \frac{v^2}{1+v}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{v^2}{1+v} - v = \frac{\cancel{v^2} - v - \cancel{v^2}}{1+v}$$

$$\Rightarrow \frac{1+v}{v}dv = -\frac{1}{x}dx$$ (By separating the variables)

$$\Rightarrow \int \frac{1+v}{v}dv = \int -\frac{1}{x} + c$$

$$\Rightarrow \int \frac{1}{v}dv + \int 1 dv = -\int \frac{1}{x} + c$$

$$\Rightarrow \log v + v = -\log x + c \Rightarrow v + \log v + \log x = c$$

$$\Rightarrow v + \log vx = c \quad [\text{since } v = y/x]$$

$$\Rightarrow \boxed{y/x + \log y = c}$$

<u>Equations reducible to the homogeneous form</u>

The equation of the form,

$$\frac{dy}{dx} = \frac{ax+by+c}{Ax+By+C}$$

where a, b, c, A, B and C are constants such that $aB \neq bA$ (i.e. $\frac{a}{A} \neq \frac{b}{B}$), can be reduced to homogeneous form by putting

$$x = X + h, \quad y = Y + k$$

where h, k being constants to be determined

$x = X+h$, $y = Y+k \Rightarrow dx = dX$, $dy = dY \Rightarrow \frac{dy}{dx} = \frac{dY}{dX}$

$$\left[\text{Now } \frac{dy}{dx} = \frac{dy}{dY}\frac{dY}{dX}\frac{dX}{dx} = 1\cdot\frac{dY}{dX}\cdot 1 = \frac{dY}{dX}\right]$$

The equation takes the form,

$$\frac{dY}{dX} = \frac{a(X+h) + b(Y+k) + c}{A(X+h) + B(Y+k) + C}$$

$$\frac{dY}{dX} = \frac{(aX+bY) + (ah+bk+c)}{(AX+BY) + (Ah+Bk+C)}$$

If we choose h and K in such away that

$$ah+bK+c=0$$
$$Ah+BK+C=0$$

then the equation reduces to

$$\frac{dY}{dX} = \frac{aX+bY}{AX+BY}$$

which is a homogeneous equation, which can be Solved by putting $Y=vX$.

Note: If $\frac{a}{A}=\frac{b}{B}$, then the above method fails to Solve the equation.

If ; $\frac{a}{A}=\frac{b}{B}=\lambda$ (say) $\Rightarrow a=A\lambda,\ b=B\lambda$

The given equation becomes,

$$\frac{dy}{dx} = \frac{\lambda(Ax+By)+c}{Ax+By+C}$$

Solving of such differential equation is delt under the section equations reducible to variable separable form - By putting $Ax+By=t$.

① Solve: $(x-4y-9)dx+(4x+y-2)dy=0$

Soln We have $(4x+y-2)dy = -(x-4y-9)dx$

$$\therefore \frac{dy}{dx} = \frac{-x+4y+9}{4x+y-2} \quad -(1)$$

[Put $x=X+h$ and $y=Y+K$, then $\frac{dy}{dx}=\frac{dY}{dX}$]

or

Put $x=X+h$ and $y=Y+K$ where h and K are constants to be chosen suitably later.

Now $\frac{dy}{dx} = \frac{dy}{dY}\frac{dY}{dX}\frac{dX}{dx} = 1.\frac{dY}{dX}.1$ Hence $\frac{dy}{dx}=\frac{dY}{dX}$

Thus (1) becomes,

$$\frac{dY}{dX} = \frac{-(X+h)+4(Y+K)+9}{4(X+h)+(Y+K)-2}$$

ie., $$\frac{dY}{dX} = \frac{(-X+4Y)+(-h+4K+9)}{(4X+Y)+(4h+K-2)} \quad -(2)$$

Let us choose h and K such that

$$-h+4K+9=0 \quad \text{and} \quad 4h+K-2=0$$

Solving these equations we get, $h=1$ and $K=-2$

Thus (2) becomes $\frac{dY}{dX} = \frac{-X+4Y}{4X+Y}$

put $Y = VX \Rightarrow \frac{dY}{dX} = V + X\frac{dV}{dX}$

Now (3) becomes, $V + X\frac{dV}{dX} = \frac{-X+4VX}{4X+VX}$

i.e., $V + X\frac{dV}{dX} = \frac{X(-1+4V)}{X(4+V)}$ or $X\frac{dV}{dX} = \frac{-1+4V}{4+V} - V$

i.e., $X\frac{dV}{dX} = \frac{-1+\cancel{4V}-\cancel{4V}-V^2}{4+V}$ i.e., $X\frac{dV}{dX} = \frac{-(1+V^2)}{4+V}$

$\therefore \frac{(4+V)dV}{1+V^2} = -\frac{dX}{X}$ by separating the variables

$\Rightarrow 4\int\frac{dV}{1+V^2} + \int\frac{VdV}{1+V^2} + \int\frac{dX}{X} = C$

i.e., $4\tan^{-1}V + \frac{1}{2}\log(1+V^2) + \log X = C$

i.e., $8\tan^{-1}V + \log(1+V^2) + 2\log X = 2C$

i.e., $8\tan^{-1}V + \log[(1+V^2)X^2] = 2C$, where $V = Y/X$

$\therefore 8\tan^{-1}\frac{Y}{X} + \log\left[\left(1+\frac{Y^2}{X^2}\right)X^2\right] = 2C = K$ (say)

But $X = x-h = x-1$ and $Y = y-k = y+2$

$\therefore 8\tan^{-1}\left(\frac{y+2}{x-1}\right) + \log\left[(x-1)^2 + (y+2)^2\right] = K$

This is the required solution.

Assignment

1. Solve $\frac{dy}{dx} = \frac{x+y-1}{x-y+1}$ Ans $2\tan^{-1}\left(\frac{y-1}{x}\right) - \log[x^2+(y-1)^2] = K$

2. Solve $\frac{dy}{dx} = \frac{x+2y-3}{2x+y-3}$ Ans $(x+y-2) = K(x-y)^3$

Linear Differential Equations

A differential equation of the form,

$$\frac{dy}{dx} + Py = Q \qquad \cdots(1)$$

where P and Q are functions of x, is called a first order linear differential equation.

Working Rule.

1. Write the given equation in the form,
$\frac{dy}{dx} + Py = Q$
2. Find the integrating factor (I.F) by evaluating,
$IF = e^{\int P dx}$

3. write the solution of the equation as,

$$y(I.F) = \int Q\,(IF)\,dx + C$$

Linear equation in 'x'

An equation of the form: $\frac{dx}{dy} + Px = Q$ where P and Q are functions of y only is called a linear equation in x.

The solution can simply be written by interchanging the role of x & y in the solution obtained already for the linear equation in y.

i.e., $$x e^{\int P dy} = \int Q\, e^{\int P dy}\, dy + C.$$

or $$x(I.F) = \int Q\,(IF)\,dy + C$$

① Solve : $(x+1)\frac{dy}{dx} - y = e^x (x+1)^2$.

soln $$\frac{dy}{dx} - \frac{y}{x+1} = \frac{e^x (x+1)^{\cancel{2}}}{\cancel{x+1}}$$

$$\frac{dy}{dx} - \frac{y}{x+1} = e^x(x+1)$$ this is of the form

$\frac{dy}{dx} + Py = Q$. Here $P = -1/x+1$ & $Q = e^x(x+1)$

$$\text{Integrating Factor} = e^{\int P dx} = e^{-\int \frac{dx}{x+1}}$$
$$= e^{-\log(x+1)}$$
$$= e^{\log(x+1)^{-1}}$$
$$I.F = \frac{1}{x+1}$$

The solution is given by

$$y.(I.F) = \int Q\,(IF)\,dx + C$$

$$\frac{y}{x+1} = \int e^x \cancel{(x+1)} \times \frac{1}{\cancel{x+1}}\, dx + C$$

$$y/x+1 = e^x + C.$$

② $x\sin x \frac{dy}{dx} + (x\cos x + \sin x)y = \sin x$

soln $$\frac{dy}{dx} + \frac{(x\cos x + \sin x)}{x\sin x}\, y = \frac{\sin x}{x\sin x}$$

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$$\frac{dy}{dx} + (\cot x + 1/x) y = \frac{1}{x}$$

which is a linear differential equation with y as dependent variable

Here $P = \cot x + 1/x$, $Q = 1/x$

$$I.F = e^{\int P dx} = e^{\int(\cot x + 1/x) dx} = e^{\log \sin x + \log x}$$

$$I.F = e^{\log x \sin x} = x \sin x$$

The solution is given by

$$y (I.F) = \int Q (I.F) dx + C$$

$$y x \sin x = \int \frac{1}{\cancel{x}} \cancel{x} \sin x dx + C$$

$$xy \sin x = -\cos x + C \quad \text{or} \quad y(x \sin x) = -\cos x + C$$

$$y(x \sin x) + \cos x = C$$

3. Solve : $x \frac{dy}{dx} - 2y = 2x$

Soln The given equation can be written as

$$\frac{dy}{dx} - \frac{2}{x} y = 2 \quad \text{This is of the form} \quad \frac{dy}{dx} + Py = Q$$

Here $P = -\frac{2}{x}$ & $Q = 2$

$$e^{\int P dx} = I.F = e^{\int -2/x \, dx} = e^{-2 \log x} = e^{\log x^{-2}} = 1/x^2$$

The general solution is given by

$$y.(I.F) = \int Q (I.F) dx + C$$

$$\frac{y}{x^2} = \int + \frac{2}{x^2} dx + C$$

$$y/x^2 = -2/x + C \Rightarrow y = Cx^2 - 2x$$

4. $\sin x \cos x \frac{dy}{dx} = y + \sin x$

Soln

$$\frac{dy}{dx} - \frac{y}{\sin x \cos x} = \frac{1}{\cos x}$$

Here $P = \frac{-1}{\sin x \cos x}$ $Q = \frac{1}{\cos x} = \sec x$

$$I.F = e^{\int P dx} = e^{\int - 1/\sin x \cos x \, dx} = e^{-\log(\tan x)} = e^{\log \cot x}$$

$$I.F = \cot x$$

$$\left[\int P dx = -\int \frac{1}{\sin x \cos x} dx \quad \div \text{ num \& den by } \cos^2 x \right.$$

$$\left. = -\int \frac{1/\cos^2 x}{\frac{\sin x \cancel{\cos x}}{\cos^{\cancel{2}} x}} dx = -\int \frac{\sec^2 x}{\tan x} dx = -\log \tan x \right]$$

$$y.(I.F) = \int Q(I.F)dx + C$$

$$y \cot x = \int \sec x \cot x\,dx + C = \int \frac{1}{\cancel{\cos x}} \frac{\cancel{\cos x}}{\sin x} dx + C$$

$$\Rightarrow y \cot x = \int \operatorname{cosec} x\,dx + C = \log(\operatorname{cosec} x - \cot x) + C.$$

5. $(1+x^2)\frac{dy}{dx} + y = e^{\tan^{-1}x}$

Soln

$$\frac{dy}{dx} + \frac{1}{1+x^2}y = \frac{e^{\tan^{-1}x}}{1+x^2}.$$

$$P = \frac{1}{1+x^2} \quad Q = \frac{e^{\tan^{-1}x}}{1+x^2}.$$

$$\int P\,dx = \int \frac{1}{1+x^2}dx = \tan^{-1}x.$$

$$I.F = e^{\int P dx} = e^{\tan^{-1}x}.$$

$$y.(I.F) = \int Q\,(I.F)\,dx + C = \int \frac{e^{\tan^{-1}x}}{1+x^2} \cdot e^{\tan^{-1}x}dx + C$$

$$y.e^{\tan^{-1}x} = \int \frac{e^{2\tan^{-1}x}}{1+x^2}dx + C.$$

consider

$$I = \int \frac{e^{2\tan^{-1}x}}{1+x^2}.dx$$

put $\tan^{-1}x = t$

$$\frac{1}{1+x^2}dx = dt$$

$$\therefore \quad I = \int e^{2t}dt = \frac{1}{2}e^{2t} = \frac{1}{2}e^{2\tan^{-1}x}$$

Thus the solution is

$$y.e^{\tan^{-1}x} = \frac{e^{2\tan^{-1}x}}{2} + C.$$

6. Solve $(1+x^2)dy + (y - \tan^{-1}x)dx = 0$

Soln The given equation can be written as

$$\frac{dy}{dx} + \frac{y - \tan^{-1}x}{1+x^2} = 0$$

$$\Rightarrow \frac{dy}{dx} + \frac{y}{1+x^2} = \frac{\tan^{-1}x}{1+x^2} \qquad P = \frac{1}{1+x^2},\ Q = \frac{\tan^{-1}x}{1+x^2}$$

$$\int P\,dx = \int \frac{1}{1+x^2}dx = \tan^{-1}x \qquad I.F = e^{\int P dx} = e^{\tan^{-1}x}$$

$\therefore$ Thus the general solution is

$$y(I.F) = \int Q\,(I.F)\,dx + C$$

$$y \cdot e^{\tan^{-1}x} = \int \frac{\tan^{-1}x}{1+x^2} \cdot e^{\tan^{-1}x} dx + C$$

Let $I = \int \tan^{-1}x \, e^{\tan^{-1}x} \frac{dx}{1+x^2}$

Put $\tan^{-1}x = t$

$\Rightarrow \frac{1}{1+x^2} dx = dt$

$\therefore I = \int t e^t dt = te^t - \int e^t dt$

$= te^t - e^t = (\tan^{-1}x - 1)e^{\tan^{-1}x}$

Thus the solution is

$$y \cdot e^{\tan^{-1}x} = (\tan^{-1}x - 1)e^{\tan^{-1}x} + C$$

7. $dx + 2xy\,dy = ye^{-y^2} dy$

Soln $dx = (ye^{-y^2} - 2xy)dy$

$\frac{dx}{dy} = ye^{-y^2} - 2xy$ or $\frac{dx}{dy} + 2y \cdot x = ye^{-y^2}$

This is a linear equation with x as dependent variable with $P = 2y$ & $Q = ye^{-y^2}$

now $I.F = e^{\int 2y\,dy} = e^{2y^2/2} = e^{y^2}$

Thus the general solution is

$x \cdot e^{y^2} = \int y e^{-y^2} e^{y^2} dy + C$

$\Rightarrow xe^{y^2} = \int y\,dy + C$

or $xe^{y^2} = \frac{y^2}{2} + C$

which is the required solution

8. Solve: $(x + \tan y)dy = \sin 2y\,dx$

Soln We have $\frac{dx}{dy} = \frac{x + \tan y}{\sin 2y}$

i.e., $\frac{dx}{dy} = x/\sin 2y + \frac{\tan y}{\sin 2y}$

$\frac{dx}{dy} - \frac{x}{2\sin y\cos y} = \frac{\tan y}{\sin 2y}$

This equation is of the form $\frac{dx}{dy} + Px = Q$,

where $P = -\frac{1}{2\sin y\cos y}$ & $Q = \frac{\tan y}{\sin 2y} = \frac{\cancel{\sin y}}{\cos y \, 2\cancel{\sin y}\cos y}$

$Q = \frac{1}{2}\sec^2 y$

$\therefore e^{\int P dy} = e^{-\frac{1}{2}\int \frac{\sec y}{\sin y} dy}$

$= e^{-1/2 \int \frac{\sec^2 y}{\tan y}} = e^{-\frac{1}{2}\log(\tan y)} = 1/\sqrt{\tan y}$

$\boxed{\int \frac{f'(x)}{f(x)} dx = \log f(x)}$

The solution is $xe^{\int P dy} = \int Q e^{\int P dy} dy + C$

i.e., $\frac{x}{\sqrt{\tan y}} = \int \frac{1}{2}\sec^2 y \frac{1}{\sqrt{\tan y}}\,dy + c$

put $\tan y = t \quad \therefore \sec^2 y\,dy = dt$

Hence $\frac{x}{\sqrt{\tan y}} = \frac{1}{2}\int \frac{dt}{\sqrt{t}} + c = \frac{1}{\cancel{2}} \frac{t^{1/2}}{\cancel{1/2}} + c$

Or

$\int \frac{1}{2}\frac{\sec^2 y}{\sqrt{\tan y}}\,dy$

$\int \frac{f'(x)}{\sqrt{f(x)}}\,dx = 2\sqrt{f(x)}$

$= \frac{1}{\cancel{2}}\cancel{2}\sqrt{\tan y}$

$= \sqrt{\tan y}$

$\rightarrow$ i.e. $\frac{x}{\sqrt{\tan y}} = \sqrt{\tan y} + c$

or $x = \tan y + c\sqrt{\tan y}$, is the required solution or $x\sqrt{\cot y} = \sqrt{\tan y} + c$

9. Solve: $x\log x\,dy = (2\log x - y)\,dx$

Soln We have $\frac{dy}{dx} = \frac{2\log x - y}{x\log x}$

or $\frac{dy}{dx} + \frac{y}{x\log x} = \frac{2}{x}$ is linear in y of the form $\frac{dy}{dx} + Py = Q$, where $P = \frac{1}{x\log x}$ and $Q = 2/x$

Hence $\int P\,dx = \int \frac{1}{x\log x}\,dx = \int \frac{1/x}{\log x}\,dx = \log(\log x)$

$\therefore$ (I.F) $= e^{\int P\,dx} = e^{\log(\log x)} = \log x$

The solution is $y\,(\text{I.F}) = \int Q\,(\text{I.F})\,dx + C$

i.e., $y\log x = \int \frac{2}{x}\cdot\log x\,dx + C$

put $\log x = t \quad \therefore \frac{1}{x}dx = dt$

Hence $y\log x = \int 2t\,dt + c$ or $y\log x = t^2 + c$

Thus $y\log x = (\log x)^2 + c$, is the required solution.

10. Solve $x\cos x\frac{dy}{dx} + (\cos x - x\sin x)y = 1$

Soln $\frac{dy}{dx} + \left(\frac{\cos x - x\sin x}{x\cos x}\right)y = \frac{1}{x\cos x}$

This is of the form $\frac{dy}{dx} + Py = Q$ where

$P = \frac{\cos x - x\sin x}{x\cos x}$ & $Q = \frac{1}{x\cos x}$

$\therefore e^{\int P\,dx} = e^{\int\left(\frac{1}{x} - \tan x\right)dx} = e^{\log x - \log\sec x}$

$= e^{\log x/\sec x} = e^{\log(x\cos x)} = x\cos x$

The solution is $y e^{\int P dx} = \int Q e^{\int P dx} dx + C$

$\therefore y \cdot x\cos x = \int \frac{1}{\cancel{x\cos x}} \cdot \cancel{x\cos x}\, dx + C$

$\therefore xy\cos x = x + C$, is the required solution

Assignment

1. $\frac{dy}{dx} - \frac{2y}{x} = x + x^2$ Ans $y/x^2 = \log x + x + C$

2. $\frac{dy}{dx} + y\cot x = \cos x$ Ans $y\sin x = \frac{-\cos 2x}{4} + C$.

3. Solve $\sqrt{1-y^2}\, dx = (\sin^{-1}y - x)\, dy$ Ans:- $xe^{\sin^{-1}y} = e^{\sin^{-1}y}(\sin^{-1}y - 1) + C$.

Equations reducible to the linear form

Form(i) $f'(y)\frac{dy}{dx} + P f(y) = Q$, where P and Q are functions of x. We put $f(y) = t$

$\therefore f'(y)\frac{dy}{dx} = \frac{dt}{dx}$

The given equation becomes $\frac{dt}{dx} + Pt = Q$ which is a linear equation in t.

Similarly $f'(x)\frac{dx}{dy} + P f(x) = Q$, where P and Q are functions of y can be reduced to the linear form by putting $f(x) = t$.

① Solve $x\frac{dy}{dx} + y\log y = xye^x$.

Soln The given equation can be written as

$\frac{1}{y}\frac{dy}{dx} + \frac{1}{x}\log y = e^x$ (i.e. by dividing by xy)

This equation is of the form,

$f'(y)\frac{dy}{dx} + P \cdot f(y) = Q$ with $f(y) = \log y$.

Put $\log y = t$ $\therefore \frac{1}{y}\frac{dy}{dx} = \frac{dt}{dx}$. Thus the given equation becomes

$\frac{dt}{dx} + \frac{1}{x}t = e^x$

This is a Linear differential equation with,

$P = 1/x$ and $Q = e^x$

Now, $I.F = e^{\int 1/x\, dx} = e^{\log x} = x$.

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Thus the general solution is

$t\,(I.F) = \int Q(I.F)\,dx + c \Rightarrow t x = \int x e^{x}\,dx + c$.

$\Rightarrow t x = x e^{x} - e^{x} + c$. since $t = \log y$

$\Rightarrow x \log y = (x-1)e^{x} + c$

(2) solve: $2y \sec^2 y^2 \frac{dy}{dx} - \frac{2}{x+1} \tan y^2 = (x+1)^3$

soln put $\tan y^2 = t \Rightarrow 2y \sec^2 y^2 \frac{dy}{dx} = \frac{dt}{dx}$

The given equation becomes

$$\frac{dt}{dx} - \frac{2t}{x+1} = (x+1)^3$$

This is a linear differential equation with

$P = -\frac{2}{x+1}$ & $Q = (x+1)^3$

Now $\int p\,dx = -\int \frac{2}{x+1}\,dx = -2\log(x+1) = \log \frac{1}{(1+x)^2}$

$$I.F = e^{\int p\,dx} = 1/(1+x)^2$$

Thus the solution is

$$t \frac{1}{(1+x)^2} = \int (x+1)^3 \frac{1}{(1+x)^2}\,dx + c$$

$\Rightarrow t \cdot \frac{1}{(1+x)^2} = \int (x+1)\,dx + c \Rightarrow t \cdot \frac{1}{(1+x)^2} = \frac{x^2}{2} + x + c$

$\Rightarrow \tan y^2 = (1+x)^2 \left[\frac{x^2}{2} + x + c\right]$.

(3) Solve: $\tan y \frac{dy}{dx} + \tan x = \cos y \cos^2 x$.

soln Dividing the given equation by $\cos y$ we have

$$\sec y \tan y \frac{dy}{dx} + \sec y \tan x = \cos^2 x \rightarrow (1)$$

Now put $\sec y = t$ $\therefore$ $\sec y \tan y \frac{dy}{dx} = \frac{dt}{dx}$

Hence (1) becomes $\frac{dt}{dx} + t \tan x = \cos^2 x$.

This equation is of the form $\frac{dt}{dx} + pt = Q$, where we have $p = \tan x$ & $Q = \cos^2 x$.

$\therefore e^{\int p\,dx} = e^{\int \tan x\,dx} = e^{\log(\sec x)} = \sec x$

The solution is $t e^{\int p\,dx} = \int Q e^{\int p\,dx}\,dx + c$

i.e., $\sec y \sec x = \int \cos^2 x \cdot \sec x\,dx + c$

$\sec y \sec x = \int \cos x\,dx + c$

Thus $\sec y \sec x = \sin x + c$, is the required solution.

Assignment

1. $e^y\left(\frac{dy}{dx}+1\right)=e^x$ Ans $e^{x+y}=\frac{e^{2x}}{2}+c$

Form(ii): $\frac{dy}{dx}+Py=Qy^n$, where P and Q are functions of x. This equation is called as Bernoulli's equation in y.

We first divide the equation throughout by y^n to obtain $\frac{1}{y^n}\frac{dy}{dx}+Py^{1-n}=Q$

put $y^{1-n}=t$ $\therefore (1-n)y^{-n}\frac{dy}{dx}=\frac{dt}{dx}$

or $\frac{1}{y^n}\frac{dy}{dx}=\frac{1}{(1-n)}\frac{dt}{dx}$

Hence (1) becomes, $\frac{1}{(1-n)}\frac{dt}{dx}+Pt=Q$

or $\frac{dt}{dx}+(1-n)P.t=(1-n)Q$ which is a linear equation in t.

Similarly $\frac{dx}{dy}+Px=Qx^n$, where P and Q are function of y is called Bernoulli's equation in x. We first divide by x^n and later put $x^{1-n}=t$ to obtain a linear equation in t.

(1) $\frac{dy}{dx}+\frac{y}{x}=y^2x$

Soln Dividing the given equation by y^2 we have

$$\frac{1}{y^2}\frac{dy}{dx}+\frac{1}{yx}=x \quad -(1)$$

put $\frac{1}{y}=t$ $\therefore -\frac{1}{y^2}\frac{dy}{dx}=\frac{dt}{dx}$

Hence (1) becomes $-\frac{dt}{dx}+\frac{t}{x}=x$ or $\frac{dt}{dx}-\frac{t}{x}=-x$

This equation is a linear equation of the form $\frac{dt}{dx}+Pt=Q$, where $P=-1/x$ & $Q=-x$

$$\therefore e^{\int P dx}=e^{-\int 1/x\, dx}=e^{-\log x}=1/x$$

The solution is $te^{\int P dx}=\int Q e^{\int P dx}dx+c$

i.e., $t \cdot \frac{1}{x} = \int -x \cdot \frac{1}{x}\, dx + c$

Thus $\frac{1}{xy} = -x + c$, is the required solution

(2) solve: $\frac{dy}{dx} - y\tan x = \frac{\sin x \cos^2 x}{y^2}$.

Multiplying the given equation by y^2 we have,

$y^2 \frac{dy}{dx} - y^3 \tan x = \sin x \cos^2 x$

Put $y^3 = t$ $\therefore 3y^2 \frac{dy}{dx} = \frac{dt}{dx}$ or $y^2 \frac{dy}{dx} = \frac{1}{3}\frac{dt}{dx}$

Hence (1) becomes $\frac{1}{3}\frac{dt}{dx} - t\tan x = \sin x \cos^2 x$

or $\frac{dt}{dx} - 3t\tan x = 3\sin x \cos^2 x$

This equation is of the form

$\frac{dt}{dx} + pt = Q$, where $P = -3\tan x$ and

$Q = 3\sin x \cos^2 x$ $\therefore e^{\int P dx} = e^{\int -3\tan x dx} = e^{-3\log(\sec x)}$

$= (\sec x)^{-3} = \cos^3 x$

The solution is $te^{\int P dx} = \int Q e^{\int P dx} dx + C$

i.e., $t\cos^3 x = \int 3 \sin x \cos^2 x \cos^3 x\, dx + C$

$t\cos^3 x = 3\int \sin x \cos^5 x\, dx + c$.

put $\cos x = u$ $\therefore -\sin x\, dx = du$

$\therefore t\cos^3 x = -3\int u^5 du + C$

i.e., $t\cos^3 x = -\frac{\cancel{3}u^6}{\cancel{6}\,2} + C = -\frac{u^6}{2} + c$

Thus $(y\cos x)^3 = \frac{-\cos^6 x}{2} + C$, is the required solution.

(3) solve: $y(2xy + e^x)dx - e^x dy = 0$

The given equation can be written in the form,

$\frac{dy}{dx} = \frac{y(2xy + e^x)}{e^x} = \frac{2xy^2}{e^x} + y$

i.e., $\frac{dy}{dx} - y = \frac{2x}{e^x} y^2$. Dividing by y^2 we get,

$\frac{1}{y^2}\frac{dy}{dx} - \frac{1}{y} = \frac{2x}{e^x}$.

put $1/y = t$ $\therefore$ $-1/y^2 \frac{dy}{dx} = \frac{dt}{dx}$

Hence (1) becomes $-\frac{dt}{dx} - t = 2x/e^x$ or $\frac{dt}{dx} + t = \frac{-2x}{e^x}$

This equation is of the form $\frac{dt}{dx} + pt = Q$ where $P = 1$ and $Q = \frac{-2x}{e^x}$ $\therefore$ $e^{\int p dx} = e^x$

The solution is

$$t e^{\int P dx} = \int Q e^{\int P dx} dx + C$$

i.e., $te^x = \int \frac{-2x}{\cancel{e^x}} . \cancel{e^x} dx + C$

i.e., $te^x = \frac{-2x^2}{2} + C = -x^2 + C$

i.e., $\frac{e^x}{y} + x^2 = C$, is the required solution or $y^{-1}e^x = C - x^2$

(4) solve: $\frac{dy}{dx} + y\cot x = y^2 \sin^2 x \cos^2 x$

Soln

$$\frac{1}{y^2}\frac{dy}{dx} + \frac{1}{y}\cot x = \sin^2 x \cos^2 x \div y^2 \rightarrow (1)$$

Let $1/y = t$ $\therefore$ $-\frac{1}{y^2}\frac{dy}{dx} = \frac{dt}{dx}$

Hence (1) becomes $-\frac{dt}{dx} + t\cot x = \sin^2 x \cos^2 x$

or $\frac{dt}{dx} - t\cot x = -\sin^2 x \cos^2 x$

$P = -\cot x$ $\quad Q = -\sin^2 x \cos^2 x$

$I.F = e^{\int P dx} = e^{-\int \cot x dx} = e^{-\log \sin x} = e^{\log(\sin x)^{-1}}$

$I.F = 1/\sin x$

The solution is

$$t(I.F) = \int Q(I.F) dx + C$$

$\frac{t}{\sin x} = -\int \sin^{\cancel{2}} x \cos^2 x . \frac{1}{\cancel{\sin x}} dx + C = -\int \sin x \cos^2 x dx + C$

put $\cos x = u$ $\therefore$ $-\sin x dx = du$

$\therefore$ $t/\sin x = \int u^2 du + c = u^3/3 + C$

i.e., $\frac{1}{y\sin x} = \frac{\cos^3 x}{3} + C$; $\frac{\operatorname{cosec} x}{y} = \frac{1}{3}\cos^3 x + C$

or $\boxed{y^{-1} = \frac{1}{3}\sin x \cos^3 x + C\sin x}$

(5) solve: $x\frac{dy}{dx} + (1-x)y = x^2 y^2$

Soln The given equation can be written as

$$\frac{1}{y^2}\frac{dy}{dx} + \left(\frac{1-x}{x}\right)\frac{1}{y} = x$$

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put $\frac{1}{y} = t \Rightarrow \frac{-1}{y^2}\frac{dy}{dx} = \frac{dt}{dx}$

Thus the given equation becomes

$$-\frac{dt}{dx} + \left(\frac{1-x}{x}\right)t = x \Rightarrow \frac{dt}{dx} - \left(\frac{1-x}{x}\right)t = -x.$$

This is a linear differential equation with $P = -\left(\frac{1-x}{x}\right)$ & $Q = -x$

$\therefore$ $I.F = e^{\int P dx} = e^{-\int 1-x/x \, dx} = e^{-\int(\frac{1}{x}-1)dx} = e^{-\log x + x}$

$= e^{x} \cdot e^{-\log x} = x^{-1}e^{x} = \frac{e^x}{x}$

The solution is

$$t \cdot \underset{(I.F)}{\frac{1}{x}e^{x}} = \int Q\,(I.F)\,dx + c$$

$$\frac{1}{y}\cdot\frac{e^x}{x} = \int -\cancel{x}\,\frac{e^x}{\cancel{x}}\,dx + c = -e^x + c$$

$$\frac{e^x}{xy} = -e^x + c \Rightarrow e^x = -e^x xy + cxy$$

$$\Rightarrow e^x(1+xy) = cxy.$$

(6) solve $\frac{dy}{dx} + \frac{y}{x-1} = xy^{1/3}$

Soln The given equation can be written as

$$\frac{1}{y^{1/3}}\frac{dy}{dx} + \frac{y^{2/3}}{x-1} = x$$

put $y^{2/3} = t \Rightarrow 2/3\, y^{2/3-1}\frac{dy}{dx} = \frac{dt}{dx} \Rightarrow \frac{1}{y^{1/3}}\frac{dy}{dx} = \frac{3}{2}\frac{dt}{dx}.$

Thus the equation becomes

$$\frac{3}{2}\frac{dt}{dx} + \frac{t}{x-1} = x \quad \text{or} \quad \frac{dt}{dx} + \frac{2}{3}\frac{t}{x-1} = \frac{2}{3}x.$$

This is a linear equation, with $P = \frac{2}{3(x-1)}$ & $Q = \frac{2}{3}x.$

now, $\int P dx = \frac{2}{3}\int\frac{1}{x-1}dx = \frac{2}{3}\log(x-1)$

$\therefore$ $I.F = e^{\int P dx} = e^{\log(x-1)^{2/3}} = (x-1)^{2/3}$

Thus the solution is

$$t(x-1)^{2/3} = \int 2/3\, x(x-1)^{2/3}dx + C$$

$$I = \frac{2}{3}\int x(x-1)^{2/3}dx = 2/3\left[x\,\frac{3}{5}(x-1)^{5/3} - \int\frac{3}{5}(x-1)^{5/3}dx\right] + c$$

$$= \frac{2}{3}\left[\frac{3}{5}x(x-1)^{5/3} - \frac{3}{5}\frac{(x-1)^{8/3}}{\frac{8}{3}}\right] + c$$

$$= \frac{2}{3}\left[\frac{3}{5}x(x-1)^{5/3} - \frac{9}{40}(x-1)^{8/3}\right] + c$$

$$y^{2/3}(x-1)^{2/3} = \frac{2}{5}x(x-1)^{5/3} - \frac{3}{20}(x-1)^{8/3} + c$$

this is the required solution.

(7) $x\frac{dy}{dx}+y=y^2\log x$

Soln The given equation can be written as

$$\frac{1}{y^2}\frac{dy}{dx}+\frac{1}{xy}=\frac{\log x}{x}$$

Put $\frac{1}{y}=t \Rightarrow -\frac{1}{y^2}\frac{dy}{dx}=\frac{dt}{dx} \Rightarrow \frac{1}{y^2}\frac{dy}{dx}=-\frac{dt}{dx}$

Thus the given equation becomes

$$-\frac{dt}{dx}+\frac{1}{x}t=\frac{\log x}{x} \text{ or } \frac{dt}{dx}-\frac{1}{x}t=-\frac{\log x}{x}$$

This is a linear equation with $P=-1/x$ & $Q=-\frac{\log x}{x}$

now $\int P\,dx=\int -\frac{1}{x}dx=-\log x \therefore I.F=e^{-\log x}=e^{\log(x)^{-1}}$

$I.F=1/x$

Thus the solution is

$$t\cdot\frac{1}{x}=-\int\frac{\log x}{x}\cdot\frac{1}{x}dx+C$$

$$=-\int \log x\frac{1}{x^2}dx+C$$

$$=-\left[\log x\int\frac{1}{x^2}dx-\int\left[\frac{1}{x}\int\frac{1}{x^2}dx\right]dx\right]+C$$

$$=-\left[\log x\left(-\frac{1}{x}\right)-\left[\int\frac{1}{x}\left(-1/x\right)dx\right]\right]+C$$

$$=-\left[-\frac{\log x}{x}+\int\frac{1}{x^2}dx\right]+C=-\left[-\frac{\log x}{x}-\frac{1}{x}\right]+C$$

$$\frac{1}{yx}=\frac{\log x}{x}+\frac{1}{x}+C \Rightarrow y\log x+y+Cxy=1$$

$$\Rightarrow y[\log x+1+Cx]=1$$

(8) Solve $x^3\frac{dy}{dx}-x^2y=-y^4\cos x$

Soln We have $x^3\frac{dy}{dx}-x^2y=-y^4\cos x$. $\div x^3y^4$ we get

$$\frac{1}{y^4}\frac{dy}{dx}-\frac{1}{y^3x}=-\frac{\cos x}{x^3}$$

put $\frac{1}{y^3}=t \therefore -\frac{3}{y^4}\frac{dy}{dx}=\frac{dt}{dx}$ or $\frac{1}{y^4}\frac{dy}{dx}=-\frac{1}{3}\frac{dt}{dx}$

Hence (1) becomes, $-\frac{1}{3}\frac{dt}{dx}-\frac{t}{x}=-\frac{\cos x}{x^3}$

i.e., $\frac{dt}{dx}+\frac{3t}{x}=\frac{3\cos x}{x^3}$

This equation is a linear equation of the form $\frac{dt}{dx}+Pt=Q$, where $P=3/x$ and $Q=\frac{3\cos x}{x^3}$

$\therefore e^{\int P\,dx}=e^{\int(3/x)dx}=e^{3\log x}=x^3$

Solution is given by $te^{\int Pdx}=\int Qe^{\int Pdx}dx+C$

i.e., $tx^3=\int\frac{3\cos x}{\cancel{x^3}}\cancel{x^3}dx+C$

i.e., $x^3/y^3 = 3\sin x + c$, is the required
Solution.

(9) Solve : $xy(1+xy^2)\frac{dy}{dx}=1$

consider $dx/dy = xy + x^2y^3$ or $\frac{dx}{dy} - xy = x^2y^3 \div x^2$

$$\frac{1}{x^2}\frac{dx}{dy} - \frac{1}{x}y = y^3 \rightarrow (1)$$

put $\frac{1}{x} = t \quad \therefore \quad -\frac{1}{x^2}\frac{dx}{dy} = \frac{dt}{dy}$

Hence (1) becomes $-\frac{dt}{dy} - ty = y^3$ or $\frac{dt}{dy} + ty = -y^3$

This equation is of the form $\frac{dt}{dy} + pt = q$,

where $p = y$ & $q = -y^3$ $\therefore$ $e^{\int p\,dy} = e^{\int y\,dy} = e^{y^2/2}$

The solution is $te^{\int p\,dy} = \int q\,e^{\int p\,dy}dy + c$

i.e., $te^{y^2/2} = -\int y^3 e^{y^2/2} dy + c$

put $y^2/2 = u \quad \therefore \quad y\,dy = du$

Also $y^2 . y\,dy = y^2 du$ or $y^3 dy = 2u . du$

$\therefore \quad te^{y^2/2} = -2\int u e^u du + c$

i.e., $te^{y^2/2} = -2(ue^u - e^u) + c$, on integration by parts. Thus $\frac{e^{y^2/2}}{x} = 2e^{y^2/2}\left(1 - \frac{y^2}{2}\right) + c$, is the required solution.

Assignment

(1) solve : $\left(xy^2 - xe^{1/x^2}\right)dx - x^2y\,dy = 0$

Ans $y^2/x^2 = e^{1/x^2} + c$

(2) Solve : $(y\log x - 2)y\,dx = x\,dy$

Ans $1/x^2y = \frac{1}{2x^2}\left(\log x + 1/2\right) + c$

(3) Solve : $6y^2dx - x(x^3 + 2y)dy = 0$

Ans $y/x^3 = -\frac{\log y}{2} + c$.

Exact Differential Equations

Definition: The differential equation, of the form,

$$Mdx + Ndy = 0$$

where M, N are functions of x, y is called exact if there exists a function $f(x,y)$, whose perfect differential df is $Mdx + Ndy$.

Statement: The necessary and the sufficient condition for the differential equation $M(x,y)dx + N(x,y)dy = 0$ to be an exact equation is $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

Further the solution of the exact equation is given by

$$\int M dx + \int N^{(y)} dy = C$$

where, in the first term we integrate $M(x,y)$ w.r.t x keeping y fixed and $N(y)$ indicate the terms in N without x (not containing x)

① Solve: $(2x+y+1)dx + (x+2y+1)dy = 0$

Soln The given equation can be written in the form $\left[\frac{dy}{dx} = -\frac{2x+y+1}{x+2y+1}\right.$ we can also solve this problem by the method of Eqn reducible to Homogeneous form]

Let $M = 2x+y+1$ and $N = x+2y+1$

$\therefore \frac{\partial M}{\partial y} = 1$ and $\frac{\partial N}{\partial x} = 1$

since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, the given equation is exact.

The solution is $\int M dx + \int N(y) dy = C$, or

$$\int_{y\text{-constant}} M dx + \int (\text{terms of } N \text{ not containing } x)\, dy = C$$

i.e., $\int_{y\text{-cons}} (2x+y+1)\,dx + \int (2y+1)\,dy = c$

Thus $x^2+xy+x+y^2+y=c$, is the required solution

(2) Solve: $(4x+3y+1)\,dx + (3x+2y+1)\,dy = 0$

Soln Let $M = 4x+3y+1$ and $N = 3x+2y+1$

$\frac{\partial M}{\partial y} = 3$ $\qquad$ $\frac{\partial N}{\partial x} = 3$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, the given equation is exact.

The solution is $\int_{y\text{-cons}} M\,dx + \int (\text{Terms of } N \text{ not containing } x)\,dy = c$

$\int_{y\text{-const}} (4x+3y+1)\,dx + \int (2y+1)\,dy = c$

$\frac{\cancel{4}^{2} x^2}{\cancel{2}} + 3xy + x + \frac{\cancel{2}y^2}{\cancel{2}} + y = c$

$2x^2+3xy+x+y^2+y=c$

(3) Solve: $(y^3-3x^2y)\,dx - (x^3-3xy^2)\,dy = 0$

Soln Let $M = y^3-3x^2y$ and $N = -x^3+3xy^2$

$\therefore$ $\frac{\partial M}{\partial y} = 3y^2-3x^2$ $\qquad$ $\frac{\partial N}{\partial x} = -3x^2+3y^2$.

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, the given equation is exact.

The solution is $\int_{y\text{-const}} M\,dx + \int N(y)\,dy = c$

i.e., $\int_{y\text{-cons}} (y^3-3x^2y)\,dx + \int 0\,dy = c$

$\therefore$ $y^3.x - x^3y = c$.

(4) Solve: $[y(1+1/x)+\cos y]\,dx + [x+\log x - x\sin y]\,dy = 0$

Soln Let $M = y(1+1/x)+\cos y$ & $N = x+\log x - x\sin y$

$\frac{\partial M}{\partial y} = 1+1/x - \sin y$ & $\frac{\partial N}{\partial x} = 1+\frac{1}{x} - \sin y$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, the given equation is exact.

The solution is $\int M\,dx + \int N(y)\,dy = c$

$\int [y(1+1/x)+\cos y]\,dx + \int 0\,dy = c$

Thus $y(x+\log x) + x\cos y = c$, is the required solution

(5) Solve: $(y^2 e^{xy^2}+4x^3)\,dx + (2xy e^{xy^2} - 3y^2)\,dy = 0$

Soln Let $M = y^2 e^{xy^2}+4x^3$ & $N = (2xye^{xy^2}-3y^2)$

$\frac{\partial M}{\partial y} = y^2 e^{xy^2}\cdot 2xy + e^{xy^2}\,2y$, $\frac{\partial N}{\partial x} = 2xy\, e^{xy^2}\cdot y^2 + e^{xy^2}\,2y$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, the given equation is exact.

The solution is $\int M\,dx + \int N(y)\,dy = c$

i.e., $\int (y^2 e^{xy^2}+4x^3)\,dx + \int -3y^2\,dy = c$

i.e., $\frac{\cancel{y^2}\cdot e^{xy^2}}{\cancel{y^2}} + \frac{\cancel{4}x^4}{\cancel{4}} - \frac{\cancel{3}y^3}{\cancel{3}} = c$

Thus $e^{xy^2}+x^4-y^3 = c$, is the required solution

<u>Assignment</u>

(1) Solve $\frac{dy}{dx} + \frac{y\cos x + \sin y + y}{\sin x + x\cos y + x} = 0$

Ans: $y\sin x + x\sin y + xy = c$.

(2) Solve $(2+2x^2\sqrt{y})\,y\,dx + (x^2\sqrt{y}+2)\,x\,dx = 0$

Ans: $2xy + \frac{2}{3}x^3 y^{3/2} = c$.

<u>Equations reducible to the exact form</u>

<u>Integrating Factor: Type-I</u>

Suppose that, for the equation $M\,dx + N\,dy = 0$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then we take their difference.

The difference $\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}$ should be close to the expression of M or N.

if it is so, then we compute $\frac{1}{M}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)$ or $\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)$

If $\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = f(x)$ or $\frac{1}{M}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = g(y)$

then $e^{\int f(x)\,dx}$ or $e^{-\int g(y)\,dy}$ is an integrating factor.

① Solve: $(4xy+3y^2-x)\,dx + x(x+2y)\,dy = 0$

Soln Let $M = 4xy+3y^2-x$ & $N = x(x+2y) = x^2+2xy$

$$\frac{\partial M}{\partial y} = 4x+6y \quad \text{and} \quad \frac{\partial N}{\partial x} = 2x+2y. \text{(The equation is not exact)}.$$

Consider $\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 4x+6y-2x-2y = 2x+4y = 2(x+2y)$ close to N.

Now $\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{2(x+2y)}{x(x+2y)} = \frac{2}{x} = f(x)$

Hence $e^{\int f(x)dx}$ is an integrating factor.

i.e., $e^{\int 2/x\, dx} = e^{2\log x} = e^{\log(x^2)} = x^2$

Multiplying the given equation by x^2 we now have,

$$M = 4x^3y + 3x^2y^2 - x^3 \quad \text{and} \quad N = x^4 + 2x^3y$$

$$\frac{\partial M}{\partial y} = 4x^3 + 6x^2y \quad \text{and} \quad \frac{\partial N}{\partial x} = 4x^3 + 6x^2y$$

Solution of the exact equation is $\int M dx + \int N(y)dy = C$

i.e., $\int (4x^3y + 3x^2y^2 - x^3)\,dx + \int 0\,dy = C$

Thus $x^4y + x^3y^2 - \frac{x^4}{4} = C$, is the required solution.

② Solve: $y(2x-y+1)\,dx + x(3x-4y+3)\,dy = 0$

Soln Let $M = y(2x-y+1)$ and $N = x(3x-4y+3)$

i.e., $M = 2xy - y^2 + y$ & $N = 3x^2 - 4xy + 3x$

$$\frac{\partial M}{\partial y} = 2x - 2y + 1, \quad \frac{\partial N}{\partial x} = 6x - 4y + 3$$

$$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 2x-2y+1-(6x-4y+3) = -4x+2y-2 = -2(2x-y+1)$$ near to M

Now, $\frac{1}{M}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{-2(2x-\cancel{y}+1)}{y(2x-\cancel{y}+1)} = \frac{-2}{y} = g(y)$

Hence I.F $= e^{-\int g(y)dy} = e^{\int 2/y\, dy} = e^{2\log y} = e^{\log(y^2)} = y^2$

Multiplying the given equation with y^2 we now have,

$$M = 2xy^3 - y^4 + y^3 \quad \text{and} \quad N = 3x^2y^2 - 4xy^3 + 3xy^2$$

$$\frac{\partial M}{\partial y} = 6xy^2 - 4y^3 + 3y^2 \;\&\; \frac{\partial N}{\partial x} = 6xy^2 - 4y^3 + 3y^2$$

The solution is $\int M dx + \int N(y)dy = C$

i.e., $\int (2xy^3 - y^4 + y^3)\,dx + \int 0\,dy = C$

Thus $x^2y^3 - xy^4 + xy^3 = C$, is the required solution.

Assignment

① solve: $(x^2+y^2+x)dx + xy\,dy = 0$

Ans $\frac{x^4}{4} + \frac{x^2y^2}{2} + \frac{x^3}{3} = c$

② solve: $y(2xy+1)dx - x\,dy = 0$ Ans $x^2 + x/y = c$.

Integrating Factor: Type-2

If the given equation $Mdx + Ndy = 0$ is of the form $y\,f(xy)\,dx + x\,g(xy)\,dy = 0$ then $\frac{1}{Mx-Ny}$ is an integrating factor provided $Mx - Ny \neq 0$

① solve: $y(xy+1)dx - x(xy-1)dy = 0$

soln The given equation is of the form:

$y f(xy)dx + x\,g(xy)\,dy = 0$ where

$M = y f(xy) = xy^2 + y$ & $N = x\,g(xy) = x - x^2y$

Now $Mx - Ny = x^2y^2 + \cancel{xy} - \cancel{xy} + x^2y^2 = 2x^2y^2$

$\therefore \frac{1}{Mx-Ny} = \frac{1}{2x^2y^2}$ is the integrating factor.

Multiplying the given equation with $1/2x^2y^2$ it becomes an exact equation where we now have.

$M = \frac{1}{2x^2y^2}(xy^2+y)$ & $N = \frac{1}{2x^2y^2}(x - x^2y)$

i.e., $M = \frac{1}{2x} + \frac{1}{2x^2y}$ & $N = \frac{1}{2xy^2} - \frac{1}{2y}$

The solution is $\int M dx + \int N(y)dy = c$

i.e., $\int\left(\frac{1}{2x} + \frac{1}{2x^2y}\right)dx + \int -\frac{1}{2y}dy = c$

i.e., $\frac{1}{2}\log x - \frac{1}{2xy} - \frac{1}{2}\log y = c$

Or $\log(x/y) - 1/xy = 2c$

② solve: $y(xy + 2x^2y^2)dx + x(xy - x^2y^2)dy = 0$

soln The equation of the form

$yf(xy)dx + x\,g(xy)\,dy = 0$ where

$M = xy^2 + 2x^2y^3$ & $N = x^2y - x^3y^2$

Now $Mx - Ny = \cancel{x^2y^2} + 2x^3y^3 - \cancel{x^2y^2} + x^3y^3 = 3x^3y^3$

Thus $1/3x^3y^3$ is the I.F. Multiplying the given equation by this IF we have an exact equation where we now have

$$M = \frac{1}{3x^2y} + \frac{2}{3x} \quad \& \quad N = \frac{1}{3xy^2} - \frac{1}{3y}$$

The solution is $\int M dx + \int N(y) dy = C$

i.e., $\int \left(\frac{1}{3x^2y} + \frac{2}{3x}\right) dx + \int -\frac{1}{3y} dy = C$

i.e., $-\frac{1}{3xy} + \frac{2}{3}\log x - \frac{1}{3}\log y = C$

or $-\frac{1}{xy} + \log x^2 - \log y = 3C$

Thus $\log(x^2/y) - \frac{1}{xy} = 3C$, is the required solution

Assignment

Solve:

① $y(1+xy+x^2y^2)dx + x(1-xy+x^2y^2)dy = 0$

Ans $\log(x/y) + xy - 1/xy = 2C$, is

* If M and N are homogeneous functions of the same degree and $Mx+Ny \neq 0$, then $\frac{1}{Mx+Ny}$ is an integrating factor for $Mdx+Ndy=0$

① $xy\,dx - (x^2+2y^2)dy = 0 \rightarrow$ ①

Soln $M = xy \quad N = -(x^2+2y^2)$

$$\frac{\partial M}{\partial y} = x \qquad \frac{\partial N}{\partial x} = -2x$$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

$\therefore$ equation ① is not an exact differential equation

Clearly M and N are homogeneous of same degree. Now $Mx+Ny = x^2y - x^2y - 2y^3 \neq 0$

$\therefore \frac{1}{Mx+Ny} = -\frac{1}{2y^3}$ is an I.F

Multiplying the equation by $-1/2y^3$, we get

$$\frac{-x}{2y^2}dx + \left(\frac{x^2}{2y^3} + \frac{1}{y}\right)dy = 0$$

This equation is exact.

$\int M dx + \int N(y) dy = C$

$-\int x/2y^2\, dx + \int \frac{1}{y} dy = C \Rightarrow -\frac{1}{2y^2}\frac{x^2}{2} + \log y = C$

$-x^2/4y^2 + \log y = C$

Type 3:- If the given equation $Mdx + Ndy = 0$ is of the form
$$x^{k_1} y^{k_2} (c_1 y\,dx + c_2 x\,dy) + x^{k_3} y^{k_4} (c_3 y\,dx + c_4 x\,dy) = 0$$
where k_i and c_i $(i = 1 \text{ to } 4)$ are constant, then $x^a y^b$ is an integrating factor. The constant a & b are determined such that the condition for an exact equation is satisfied

① Solve: $x(3y\,dx + 2x\,dy) + 8y^4(y\,dx + 3x\,dy) = 0$

Soln The given equation can be rearranged as
$$(3xy + 8y^5)dx + (2x^2 + 24xy^4)dy = 0 \rightarrow ①$$
Multiplying ① by $x^a y^b$ we have
$$M = 3x^{a+1} y^{b+1} + 8x^a y^{b+5}$$
$$N = 2x^{a+2} y^b + 24x^{a+1} y^{b+4}$$
We shall find a & b such that $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$
$$\therefore \quad 3x^{a+1}(b+1)y^b + 8x^a(b+5)y^{b+4} = 2y^b(a+2)x^{a+1} + 24y^{b+4}(a+1)x^a$$
$$\Rightarrow \quad 3(b+1) = 2(a+2) \text{ and } 8(b+5) = 24(a+1)$$
or $\quad 2a - 3b = -1 \quad$ & $\quad 3a - b = 2$

By solving we get $a = 1$, $b = 1$.

Thus we now have $M = 3x^2y^2 + 8xy^6$

& $N = 2x^3y + 24x^2y^5$

The solution is $\int M\,dx + \int N(y)\,dy = c$

i.e., $\int (3x^2y^2 + 8xy^6)dx + \int 0\,dy = c$

Thus $x^3y^2 + 4x^2y^6 = c$ is the required solution

Assignment

1. Solve: $x(4y\,dx + 2x\,dy) + y^3(3y\,dx + 5x\,dy) = 0$

Ans $x^4y^2 + x^3y^5 = c$.

Type 4: Exactness by inspection

E_1: $dx \pm dy = d(x \pm y)$

E_2: $x\,dy + y\,dx = d(xy)$

E_3 $\quad \dfrac{x\,dy - y\,dx}{x^2} = d(y/x)$

E_4 $\quad \dfrac{y\,dx - x\,dy}{y^2} = d(x/y)$

E_5 $\quad \dfrac{x\,dy - y\,dx}{x^2 - y^2} = d\left[\frac{1}{2}\log\left(\frac{x+y}{x-y}\right)\right]$

E_6 $\quad \dfrac{x\,dy - y\,dx}{x^2 + y^2} = d\left[\tan^{-1}\left(\frac{y}{x}\right)\right] = -d\left[\tan^{-1}\left(\frac{x}{y}\right)\right]$

E_7 $\quad \dfrac{x\,dx + y\,dy}{x^2 + y^2} = d\left[\frac{1}{2}\log(x^2+y^2)\right] = d\left[\log\sqrt{x^2+y^2}\right]$

E_8 $\quad \dfrac{x\,dx + y\,dy}{\sqrt{x^2+y^2}} = d\left[\sqrt{x^2+y^2}\right]$

(1) Solve: $x\,dx + y\,dy - \dfrac{x\,dy - y\,dx}{x^2+y^2} = 0$

soln The given equation is equivalent to the form

$$x\,dx + y\,dy - d\left[\tan^{-1}(y/x)\right] = 0$$

Integrating, we get

$$\frac{x^2}{2} + \frac{y^2}{2} - \tan^{-1}(y/x) = c$$

(2) solve $\dfrac{y\,dx - x\,dy}{y^2} + (x\,dx + y\,dy) = 0$

soln The given equation is equivalent to the form,

$$d\left(\frac{x}{y}\right) + x\,dx + y\,dy = 0$$

$\Rightarrow \quad x/y + \dfrac{x^2}{2} + \dfrac{y^2}{2} = c$, on integration

thus $\quad \dfrac{x}{y} + \dfrac{1}{2}\left[x^2 + y^2\right] = c$, is the required Solution.

(3) solve: $y\,e^{x/y}\,dx = (x\,e^{x/y} + y^2)\,dy$

soln The given equation can be put in the form,

$$e^{x/y}(y\,dx - x\,dy) = y^2\,dy$$

or $\quad e^{x/y}\left(\dfrac{y\,dx - x\,dy}{y^2}\right) = dy$

i.e., $\quad e^{x/y}\,d(x/y) = dy$

Integration yields $\quad e^{x/y} = y + c$

Thus $e^{x/y} - y = c$, is the required solution

(4) Solve: $x\,dx = y(x^2+y^2-1)dy$

Soln The given equation can be put in the form

$$x\,dx + y\,dy = y(x^2+y^2)\,dy$$

$$\frac{x\,dx+y\,dy}{x^2+y^2} = y\,dy$$

$$d\left[\frac{1}{2}\log(x^2+y^2)\right] = y\,dy$$

Integrating we get, $\frac{1}{2}\log(x^2+y^2) = \frac{y^2}{2} + c$

or $\log(x^2+y^2) = y^2 + 2c = y^2 + k$ (say $k = 2c$)

Thus $\log(x^2+y^2) - y^2 = k$ is the required solution

(5) Solve: $[1+y\tan(xy)]dx + [x\tan(xy)]dy = 0$

Soln The given equation can be put in the form

$$dx + \tan(xy)\,[y\,dx + x\,dy] = 0$$

i.e. $dx + \tan(xy)\,d(xy) = 0$

Integrating we get, $x + \log\sec(xy) = c$

Assignment

(1) $x\,dx + y\,dy = 2(x^2+y^2)y\,dy$

Ans $\log(x^2+y^2) - 2y^2 = c$.

Orthogonal Trajectories

Defn:- If two family of curves are such that every member of one family intersects every member of the other family at right angles then they are said to be orthogonal trajectories of each other.

Method of finding the orthogonal trajectories.

case(i): (Cartesian family)

1. Given $f(x,y,c) = 0$, differentiate w.r.t x & eliminate c.

2. Replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ and solve the equation

case(ii): (polar family)

① Given an equation in r and θ, we prefer to take logarithms first & then differentiate w.r.t θ

② After ensuring that the given parameter is eliminated we replace $\frac{dr}{d\theta}$ by $-r^2\frac{d\theta}{dr}$ and solve the equation.

1. Find the O.T of the family of parabolas $y^2 = 4ax$.

Soln consider $\frac{y^2}{x} = 4a$

Now differentiate (1) w.r.t x we have

$$\frac{x\,2y\frac{dy}{dx} - y^2}{x^2} = 0 \quad \text{or} \quad 2xy\frac{dy}{dx} - y^2 = 0$$

i.e $2x\frac{dy}{dx} - y = 0$, is the D.E of the given family.

Replacing $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ we have,

$$2x\left(-\frac{dx}{dy}\right) - y = 0 \quad \text{or} \quad 2x\frac{dx}{dy} + y = 0$$

$\Rightarrow$ $2x\,dx = -y\,dy$ $\Rightarrow$ $2x\,dx + y\,dy = 0$

$\Rightarrow$ $\int 2x\,dx + \int y\,dy = 0$

i.e., $x^2 + \frac{y^2}{2} = c$ or $2x^2 + y^2 = 2c = k$ (say)

Thus $\boxed{2x^2 + y^2 = k}$ is the required O.T

2. Find the O.T of the family of astroids $x^{2/3} + y^{2/3} = a^{2/3}$

Soln consider $x^{2/3} + y^{2/3} = a^{2/3}$

Differentiating w.r.t x, we have

$$\frac{2}{3}x^{2/3-1} + \frac{2}{3}y^{2/3-1}\frac{dy}{dx} = 0 \quad \text{x'ly } 3/2$$

$x^{-1/3} + y^{-1/3}\frac{dy}{dx} = 0$, is the D.E of the given family

Replacing $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ we have,

$$x^{-1/3} + y^{-1/3}\left(-\frac{dx}{dy}\right) = 0 \quad \text{i.e.,} \quad x^{-1/3}dy = y^{-1/3}dx$$

i.e., $y^{1/3}dy = x^{1/3}dx$ by separating the variables

$$\Rightarrow \int y^{1/3} dy - \int x^{1/3} dx = c$$

i.e., $\dfrac{y^{4/3}}{4/3} - \dfrac{x^{4/3}}{4/3} = c$ or $x^{4/3} - y^{4/3} = -\dfrac{4c}{3} = K$ (say)

Thus $x^{4/3} - y^{4/3} = K$ is the required O.T.

3. Show that the family of curves $x^3 - 3xy^2 = c_1$ and $y^3 - 3x^2y = c_2$ are O.T of each other

Soln Let us consider $x^3 - 3xy^2 = c_1$ & differentiate w.r.t x.

$$\therefore \quad 3x^2 - 3\left(x\, 2y \frac{dy}{dx} + y^2\right) = 0$$

i.e., $x^2 - y^2 = 2xy \dfrac{dy}{dx}$, is the DE of the given family.

Replacing $\dfrac{dy}{dx}$ by $-\dfrac{dx}{dy}$ we have,

$$x^2 - y^2 = 2xy\left(-\frac{dx}{dy}\right)$$

or $\quad 2xy\,dx + (x^2 - y^2)\,dy = 0$

(This is a homogeneous equation. But it is also exact)

Let $M = 2xy$ and $N = x^2 - y^2$

$\therefore \dfrac{\partial M}{\partial y} = 2x$ and $\dfrac{\partial N}{\partial x} = 2x$. Hence the equation is exact

The solution is given by $\int M dx + \int N(y) dy = c$

i.e., $\int 2xy\,dx + \int -y^2 dy = c$

i.e., $x^2y - y^3/3 = c$ or $3x^2y - y^3 = 3c$

$\therefore \; y^3 - 3x^2y = c_2$ (say) is the required O.T where $c_2 = -3c$

Thus $x^3 - 3xy^2 = c_1$ & $y^3 - 3x^2y = c_2$ are orthogonal trajectories of each other.

(4) Find the orthogonal trajectories of the family of curves $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2+\lambda} = 1$, where λ is the parameter.

Soln We have $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2+\lambda} = 1 \rightarrow (1)$

Differentiating w.r.t x we have,

$$\frac{2x}{a^2} + \frac{2yy_1}{b^2+\lambda} = 0, \quad \text{where } y_1 = dy/dx$$

i.e., $\dfrac{x}{a^2} = \dfrac{-yy_1}{b^2+\lambda} \rightarrow (2)$

Also from (1) $\dfrac{x^2}{a^2} - 1 = \dfrac{-y^2}{b^2+\lambda}$

or $\dfrac{x^2-a^2}{a^2} = \dfrac{-y^2}{b^2+\lambda} \rightarrow (3)$

Now, dividing (2) by (3) we get,

$$\frac{x}{x^2-a^2} = \frac{yy_1}{y^2} \quad \text{or} \quad \frac{x}{x^2-a^2} = \frac{y_1}{y}$$

Now let us replace $y_1 = \dfrac{dy}{dx}$ by $-\dfrac{dx}{dy}$

$$\therefore \quad \frac{x}{x^2-a^2} = \frac{1}{y}\left(-\frac{dx}{dy}\right)$$

or $\quad y\,dy = -\left(\dfrac{x^2-a^2}{x}\right)dx$ by separating the variables

$$\Rightarrow \int y\,dy = -\int x\,dx + a^2\int \frac{dx}{x} + c$$

i.e., $\dfrac{y^2}{2} = -\dfrac{x^2}{2} + a^2\log x + c$

or $\quad x^2 + y^2 - 2a^2\log x - b = 0$ where $b = 2c$ is the required O.T.

(5) Find the Orthogonal trajectories of the family of curves $\dfrac{x^2}{a^2+\lambda} + y^2/_{b^2+\lambda} = 1$, where λ is a parameter.

OR

Show that the family of conics $\dfrac{x^2}{a^2+\lambda} + \dfrac{y^2}{b^2+\lambda} = 1$, where λ is a parameter is self orthogonal.

Soln consider $\dfrac{x^2}{a^2+\lambda} + \dfrac{y^2}{b^2+\lambda} = 1 \rightarrow (1)$

Differentiating w.r.t x we have,

$$\frac{2x}{a^2+\lambda} + \frac{2yy_1}{b^2+\lambda} = 0$$

or $\dfrac{x}{a^2+\lambda} + \dfrac{yy_1}{b^2+\lambda} = 0$ or $\dfrac{x}{a^2+\lambda} = \dfrac{-yy_1}{b^2+\lambda} \rightarrow (2)$

we have to eliminate λ to obtain the D.E of the given family.

$$x(b^2+\lambda) + yy_1(a^2+\lambda) = 0$$

$$xb^2 + \underline{x\lambda} + yy_1a^2 + \underline{yy_1\lambda} = 0$$

$$\therefore \lambda(x+yy_1) + xb^2 + yy_1a^2 = 0$$

$$\lambda(x+yy_1) = -(xb^2+yy_1a^2)$$

$$\therefore \lambda = -\frac{(xb^2+yy_1a^2)}{x+yy_1}$$

$$\therefore a^2+\lambda = a^2 - \left(\frac{xb^2+yy_1a^2}{x+yy_1}\right)$$

$$= \frac{a^2(x+yy_1) - (xb^2+yy_1a^2)}{x+yy_1}$$

$$= \frac{a^2x + \cancel{yy_1a^2} - xb^2 - \cancel{yy_1a^2}}{x+yy_1}$$

$$a^2+\lambda = x(a^2-b^2)/x+yy_1$$

$$b^2+\lambda = b^2 - \left(\frac{xb^2+yy_1a^2}{x+yy_1}\right) = \frac{\cancel{xb^2}+yy_1b^2-\cancel{xb^2}-yy_1a^2}{x+yy_1}$$

$$b^2+\lambda = -(a^2-b^2)yy_1/x+yy_1$$

Thus eqn ① becomes

$$\frac{x^{\cancel{2}}}{\frac{\cancel{x}(a^2-b^2)}{x+yy_1}} + \frac{y^{\cancel{2}}}{\frac{-(a^2-b^2)\cancel{y}y_1}{x+yy_1}} = 1$$

$$\frac{x(x+yy_1)}{a^2-b^2} - \frac{y(x+yy_1)}{(a^2-b^2)y_1} = 1$$

$$\frac{x+yy_1}{a^2-b^2}\left[x - \frac{y}{y_1}\right] = 1$$

$$(x+yy_1)\left[x-y/y_1\right] = a^2-b^2 \rightarrow (3)$$

This is the diff Equn of the given family

Replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$, i.e., replace y_1 by $-1/y_1$ we get

$$(x-y/y_1)(x+yy_1) = a^2-b^2 \rightarrow (4)$$

This is the diff eqn of the O.T of (1).

But the equations ② & ③ are identical

This shows that the given family is self Orthogonal.

⑥ Show that the family of parabolas $y^2 = 4a(x+a)$ is self orthogonal.

Soln Consider $y^2 = 4a(x+a) \rightarrow (1)$

Differentiating w.r.t x, we have

$$2y\frac{dy}{dx} = 4a \quad \therefore a = yy_1/2 \quad \text{where } y_1 = \frac{dy}{dx}$$

Substituting this value of 'a' in (1) we have,

$$y^2 = 2yy_1\left(x + yy_1/2\right) \text{ or } y = 2xy_1 + yy_1^2$$

Thus we have, $y = 2xy_1 + yy_1^2 \rightarrow (2)$

This is the D.E of the given family.

Now replacing y_1 by $-1/y_1$ (2) becomes

$$y = 2x\left(-\frac{1}{y_1}\right) + y\left(-1/y_1\right)^2 \text{ or } y = -\frac{2x}{y_1} + \frac{y}{y_1^2}$$

$$\therefore yy_1^2 + 2xy_1 = y \rightarrow (3)$$

(3). the DE of the orthogonal family which is same as (2) being the d.E. of the given family.

Thus the family of parabolas $y^2 = 4a(x+a)$ is self orthogonal.

Assignment

① Find the O.T of the family $y^2 = cx^3$

Ans $2x^2 + 3y^2 = K$ where $K = 2C$

Polar family

① Find the O.T of the family $r = a(1+\sin\theta)$

Ans We have $r = a(1+\sin\theta)$

$\Rightarrow$ $\log r = \log a + \log(1+\sin\theta)$ Diff w.r.t θ we have

$$\frac{1}{r}\frac{dr}{d\theta} = 0 + \frac{\cos\theta}{1+\sin\theta}$$

Replacing $\frac{dr}{d\theta}$ by $-r^2\frac{d\theta}{dr}$ we get,

$$\frac{1}{r}\left(-r^2\frac{d\theta}{dr}\right) = \frac{\cos\theta}{1+\sin\theta}$$

i.e., $-r\frac{d\theta}{dr} = \frac{\cos\theta}{1+\sin\theta}$

or $\frac{1+\sin\theta}{\cos\theta}\,d\theta = -\frac{dr}{r}$ by separating the variables

$\Rightarrow \quad \int \frac{dr}{r} + \int \frac{1+\sin\theta}{\cos\theta}\, d\theta = c$

i.e., $\log r + \int \sec\theta\, d\theta + \int \tan\theta\, d\theta = c$

i.e., $\log r + \log(\sec\theta + \tan\theta) + \log\sec\theta = c$

i.e., $\log[r(\sec\theta + \tan\theta)\sec\theta] = \log b$ (say)

$\Rightarrow \quad r\left(\frac{1}{\cos\theta} + \frac{\sin\theta}{\cos\theta}\right)\frac{1}{\cos\theta} = b$

$\Rightarrow \quad r\left(\frac{1+\sin\theta}{\cos^2\theta}\right) = b$ or $\frac{r(1+\sin\theta)}{1-\sin^2\theta} = b$

Thus $r = b(1-\sin\theta)$ is the required O.T.

2. Find the O.T of the family $2a/r = 1-\cos\theta$

Soln We have $2a/r = 1-\cos\theta$

$\Rightarrow \quad \log 2a - \log r = \log(1-\cos\theta)$

Differentiating w.r.t θ we have,

$$-\frac{1}{r}\frac{dr}{d\theta} = \frac{\sin\theta}{1-\cos\theta}$$

Replacing $\frac{dr}{d\theta}$ by $-r^2\frac{d\theta}{dr}$ & simplifying R.H.S we have

$$-\frac{1}{\cancel{r}}\left(-r^{\cancel{2}}\frac{d\theta}{dr}\right) = \frac{\cancel{2}\sin(\cancel{\theta/2})\cos(\theta/2)}{\cancel{2}\sin^{\cancel{2}}(\theta/2)}$$

i.e., $r\frac{d\theta}{dr} = \cot(\theta/2)$

or $\tan(\theta/2)\, d\theta = \frac{1}{r}dr$

$\therefore \quad \int \frac{1}{r}dr - \int \tan(\theta/2)\, d\theta = c$

i.e., $\log r - \frac{\log\sec(\theta/2)}{(1/2)} = c$

i.e., $\log r - 2\log\sec(\theta/2) = c$

$\log[r/\sec^2(\theta/2)] = \log b$ (say)

$\Rightarrow \quad r/\sec^2(\theta/2) = b$ or $r\cos^2(\theta/2) = b$

This is the required equation of the O.T which can be put in the following form.

$r\frac{1}{2}(1+\cos\theta) = b$ or $r(1+\cos\theta) = 2b$

$\therefore \quad 2b/r = 1+\cos\theta$ is the required O.T.

3. Find the O.T of the family of Lemniscates $r^2 = a^2\cos 2\theta$

Soln We have, $r^2 = a^2\cos 2\theta$

$\Rightarrow \quad 2\log r = 2\log a + \log(\cos 2\theta)$

Differentiating w.r.t θ we get,

$\frac{2}{r}\frac{dr}{d\theta} = \frac{-2\sin 2\theta}{\cos 2\theta}$. i.e., $\frac{1}{r}\frac{dr}{d\theta} = -\tan 2\theta$.

Replacing $\frac{dr}{d\theta}$ by $-r^2\frac{d\theta}{dr}$ we get,

$$\frac{1}{r}\left(-r^2\frac{d\theta}{dr}\right) = -\tan 2\theta \quad \text{or} \quad r\frac{d\theta}{dr} = \tan 2\theta$$

$\therefore \quad \frac{d\theta}{\tan 2\theta} = \frac{1}{r}dr$ by separating the variables.

$$\Rightarrow \int\frac{dr}{r} - \int \cot 2\theta \, d\theta = c$$

i.e., $\log r - \frac{1}{2}\log(\sin 2\theta) = c$

i.e., $\log\left[\frac{r}{\sqrt{\sin 2\theta}}\right] = \log b$ (say) $\Rightarrow r = b\sqrt{\sin 2\theta}$

Thus $r^2 = b^2\sin 2\theta$ is the required O.T

Assignment

1. Find the O.T of the family $r^n\cos n\theta = a^n$

Ans $r^n\sin n\theta = b$

2. Using the concept of orthogonal trajectories show that the family of curves $r = a(\sin\theta + \cos\theta)$ & $r = b(\sin\theta - \cos\theta)$ intersect each other orthogonally.